

SUBCONVEXITY FOR THE RANKIN-SELBERG L-FUNCTION IN BOTH LEVELS

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ABSTRACT. In this paper, we obtain a subconvexity result for the Rankin-Selberg L-function in both levels. The new feature in this result is applying an amplification method of Duke-Friedlander-Iwaniec to a double Petersson-Kuznetsov trace formula. As the trace formula ranges over all the GL_2 spectrum, this subconvexity result is unconditional of which Hecke eigenforms are chosen in the Rankin-Selberg L-function.

1. INTRODUCTION

We apply techniques gained from a beyond endoscopy approach to the Rankin-Selberg L-function in [H] to understanding subconvexity. In [H], we studied an average of Rankin-Selberg L-functions $L(f \times h, s)$. More specifically, for a smooth compactly supported test function g , we investigated

$$\sum_n g\left(\frac{n}{X}\right) \sum_h \sum_f a_n(f) a_n(h)$$

as X gets large, with $a_n(f), a_n(g)$ being weighted Fourier coefficients associated to holomorphic or Maass Hecke eigenforms f, h . On applying Mellin inversion to this formula, we are studying averages of Rankin-Selberg L-functions. Understanding this sum is reminiscent to understanding an approximate functional equation for the Rankin-Selberg L-function. However the important difference, which is explained in more detail below, is that the parameter X above has no apparent connection to the essential data (level, weight,...) of the L-function, while in the approximate function equation the parameter(s) which one investigates in subconvexity problems are intrinsically associated to X . For example, the sum

$$\sum_n \frac{1}{\sqrt{n}} g\left(\frac{n}{pq}\right) \sum_h \sum_f a_n(f) a_n(h)$$

over forms f, h of level p, q , respectively, is roughly the approximate functional equation for the Rankin-Selberg L-function. In an approach to breaking convexity in both levels, one would naively consider this sum as $p, q \rightarrow \infty$.

Given this connection of beyond endoscopy and the approximate functional equation, we ask if we can learn anything more about subconvexity for the Rankin-Selberg L-function. The clear connection of the two above sums hints we should study subconvexity as the two levels of the associated automorphic forms of the L-function go to infinity.

This question of subconvexity for the Rankin-Selberg L-function in the level aspect has been studied in [DFI], [KMV], and [MV]. However, in these cases, one of the levels remains fixed as the other goes to infinity. Recently in [HM] the first case where both levels vary was announced. We state their result. Let M be a positive square-free integer and P a prime with $(M, P) = 1$, $P \sim M^\eta$ and $0 < \eta < \frac{2}{21}$. Then for two holomorphic newforms f, h we have

the bound

$$L(f \times h, 1/2) \ll (MP)^{1/2+\epsilon} \left(\frac{1}{(MP)^{\frac{\eta}{2(1+\eta)}}} + \frac{1}{(MP)^{\frac{2-21\eta}{64(1+\eta)}}} \right).$$

We make no such restrictions in this paper and associate the subconvexity of the Rankin-Selberg L-function in both levels to subconvexity bounds of Dirichlet L-functions. Our result is:

Theorem 1.1. *Let f, h be holomorphic or Maass Hecke eigenforms associated to GL_2 automorphic representations of prime level p, q , with $p \neq q$ with nontrivial central characters χ, ψ , respectively. Then for any $\epsilon > 0$ and a fixed but arbitrarily small $\delta > 0$ as $p, q \rightarrow \infty$, we have*

$$L(f \times h, 1/2) \ll (pq)^\epsilon \left((pq)^{\frac{3\theta}{2}-1/8} + (pq)^{3/8+\frac{\theta}{4}} + p^{3/8+\frac{\theta}{4}} q^{\frac{\theta}{4}-1/8} + q^{3/8+\frac{\theta}{4}} p^{\frac{\theta}{4}-1/8} + (1+q^\delta)(pq)^{\frac{3}{8}+\frac{\theta}{4}} \right).$$

Here $\theta > 0$ comes from a (sub)convexity bound for the Dirichlet L-functions $L(\chi, 1/2) \ll p^{\theta+\epsilon}$ and $L(\psi, 1/2) \ll q^{\theta+\epsilon}$.

Remark. The convexity bound for the Rankin-Selberg L-function in the level aspect is

$$L(f \times h, 1/2) \ll (pq)^{\frac{1}{2}+\epsilon}.$$

Corollary 1.2. *If χ and ψ are real quadratic characters, then we can apply Theorem 1.1 with $\theta = \frac{1}{6}$ using the result of [CI] to get the bound*

$$L(f \times h, 1/2) \ll (pq)^{\frac{10}{24}+\epsilon}.$$

If one of the characters is not real then we can apply the bound $\theta = \frac{3}{16}$ of [B] to get

$$L(f \times h, 1/2) \ll (pq)^{\frac{27}{64}+\epsilon}.$$

2. DETAILS OF PAPER

Inspired by [DFI] and [KMV], we apply the amplification method. Ignoring convergence issues for now, we study

$$(2.1) \quad \sum_{f \in H(p, \chi)} \left| \sum_{l \leq L} x_l \lambda_f(l) \right|^2 \sum_{h \in H(q, \psi)} |L(f \times h, 1/2)|^2,$$

where x_l are arbitrary complex coefficients which we optimize to “amplify” a single Rankin-Selberg L-function. Using an approximate functional equation and then a double Kuznetsov trace formula similar to [H], we bound (2.1) in terms of a function F of p, q , and L . In order to bound just a single L-function, we need positivity to get to an estimate

$$\left| \sum_{l \leq L} x_l \lambda_f(l) \right|^2 \sum_{h \in H(q, \psi)} |L(f \times h, 1/2)|^2 \ll F(p, q, x_l, L).$$

Then choosing the x_l to make $|\sum_{l \leq L} x_l \lambda_f(l)|^2$ as large as possible ensures that

$$L(f \times h, 1/2) \ll \left(\frac{F(p, q, x_l, L)}{|\sum_{l \leq L} x_l \lambda_f(l)|} \right)^{1/2}$$

is as small as possible.

To the details of actually getting the bound $F(p, q, x_l, L)$ after applying the double Kuznetsov formula, we open up the Kloosterman sums and reorganize the c_1, c_2 -sums and associated exponential sums to get terms more amenable to estimation similar to [H].

One then has four sums needing estimation. Two come from the geometric sides of the trace formulas (labelled c_1, c_2), and the two sums from opening up $|L(f \times h, s)|^2$ with the approximate functional equation. Performing Poisson summation and Mellin inversion at various steps reduces the estimation to GL_1 Dirichlet L-functions.

Another feature in the calculation is that one sees a shadow of the Selberg trace formula. We motivate this connection with a short heuristic. First, we denote by $H(p, \chi)$ the automorphic forms with level p and central character χ . Ignoring test functions and convergence issues, the second moment calculation essential to this paper is:

$$(2.2) \quad \sum_{f \in H(p,1), h \in H(q,1)} |L(1/2, f \times h)|^2 \approx \sum_{f \in H(p,1), h \in H(q,1)} \sum_{m,n \geq 1} \frac{1}{(mn)^{1/2}} g\left(\frac{n}{pq}\right) g'\left(\frac{m}{pq}\right) a_f(n) a_f(m) a_h(n) a_h(m) \approx \left(\frac{6\sqrt{pq}}{\pi^2} \int_{-\infty}^{\infty} \frac{g(t)}{t^{1/2}} dt \right) \sum_{m \geq 1} \frac{1}{m^{1/2}} g'\left(\frac{m}{pq}\right) \left(\sum_{F \in H((p,q),1)} |a_F(m)|^2 \right).$$

The second approximation is just using classical Rankin-Selberg theory on the n -sum, or one can also apply an analogous argument of [H]. Another application of Rankin-Selberg theory to the m -sum gives as a main term for (2.2):

$$(2.3) \quad \frac{6pq}{\pi^2} \tilde{g}(1/2) \tilde{g}'(1/2) \left(\sum_{F \in H((q,D),1)} \text{Res}_{s=1} L(1, F \times \overline{F}) \right)$$

with \tilde{g} denoting the Mellin transform of g . But this term $\text{Res}_{s=1} L(1, F \times \overline{F})$ cancels the orthonormalization built into the Kuznetsov trace formula. This final formula is an unweighted trace formula, or a Selberg trace formula. Though in this paper we take nontrivial central characters $\chi \bmod (p), \psi \bmod (q)$, something similar to this phenomenon is still seen. Fortunately recognizing the geometric side of the Selberg trace formula, in this fashion, has been encountered and studied by Rudnick in his thesis [R]. There also seems to be some connection of interactions of the geometric sides of the Kuznetsov and Selberg trace formula in the analysis in [SY], which counts the number of closed geodesics up to a large parameter X .

Next, though we explicitly avoided Bessel functions in our initial Kuznetsov trace formulas—in estimating the archimedean integrals that occur in our estimation—we do see Bessel functions. However, they display themselves “late” in the calculation, and we use standard asymptotics to deal with them. We say “late” in the sense that one could approach the above kind of subconvexity problem in Theorem 1.1 using one Kuznetsov trace formula (for the f -sum) and one Voronoi summation for a specific automorphic form h , very similar to [KMV]. In the early stages of such a calculation (specifically after Voronoi summation), one encounters difficult arithmetic and Bessel function analysis. Another difficulty is that one would like to understand subconvexity in both levels for all GL_2 forms, so one would need a different Voronoi summation for each type of automorphic form h (holomorphic, Maass, and Eisenstein). Doing a double Kuznetsov trace formula simplifies this analysis. In another paper [H1], we prove that the Kuznetsov (Petersson) trace formula implies Voronoi summation, so no information is lost in the approach of the current paper.

Finally, in order to ensure positivity of

$$(2.4) \quad \sum_{f \in H(p, \chi)} h(V, t_f) \left| \sum_{l \leq L} x_l \lambda_f(l) \right|^2 \sum_{h \in H(q, \psi)} h(W, t_h) |L(1/2, f \times h)|^2,$$

—where now we include the test functions that ensure convergence (as compared to (2.1))—in section 10 we discuss how we can choose V, W such that the test functions $h(W, t_h), h(V, t_f)$ are “large” for automorphic forms with eigenvalues (or weights) $\{t_{f^\circ}, t_{h^\circ}\}$ and arbitrarily small at any other spectral values.

2.1. The Kloosterman-Kloosterman term. When we apply the trace formula to both spectral sums, we encounter four different terms: the delta-delta term, the delta-Kloosterman term, Kloosterman-delta term, and the Kloosterman-Kloosterman term. The first three are relatively straightforward to deal with. However, the Kloosterman-Kloosterman term is much more difficult, and we explain the steps in making an estimate on it.

This term comes from after applying the Kuznetsov trace formula twice and is seen in the total geometric side of the trace formula in (5.6),

$$(2.5) \quad \sqrt{pq} \sum_{m \geq 1} \frac{1}{m^{1/2}} F'_{M'}\left(\frac{d^\circ m}{pq}\right) \sum_{\substack{c_1 \equiv 0(p) \\ c_2 \equiv 0(q)}} \frac{1}{c_1 c_2} \sum_{x(c_1)^*} \sum_{y(c_2)^*} \chi(\bar{x}) \psi(\bar{y}) e\left(\frac{\bar{x} d^\circ m}{c_1} + \frac{\bar{y} \frac{l_1 l_2 m}{d^\circ k^2}}{c_2}\right) \times \\ \left\{ \sum_k \int_{-\infty}^{\infty} e\left(pqt\left(\frac{c_2 x + c_1 y - k c_1 c_2}{c_1 c_2}\right)\right) \frac{F_M(t)}{t^{1/2}} V\left(\frac{4\pi \sqrt{tpq d^\circ m}}{c_1}\right) W\left(\frac{4\pi \sqrt{tpq \frac{l_1 l_2 m}{d^\circ k^2}}}{c_2}\right) dt. \right.$$

The functions $F_M, F'_{M'}$ come from a partition of unity of the approximate functional equation of the Rankin-Selberg function. The functions W, V come from our choice of test functions in the trace formula. Trivially (2.5) has a bound of $(pq)^2$, we will sketch how to get the bound $\frac{p^{\theta_1 + \epsilon} q^{\theta_2 + \epsilon}}{\sqrt{pq}}$, where $\theta_1, \theta_2 \leq 1/4$.

Fix the m -sum for now and look at the c_1, c_2 , and k -sums. We will now rearrange terms following [H]. Naively, the idea is for each integer n finding all c_1, c_2, x, y, k such that $n = c_2 x + c_1 y - k c_1 c_2$. Rearranging the sums in this way, we can write $\bar{x}(c_1)$ and $\bar{y}(c_2)$ in terms of a parameter $r(n)$. Specifically, for $x(c_1)$ we have $\bar{x} = \frac{c_1 r + c_2}{n}(c_1)$. The advantage of this is that the difficult Kloosterman sums are essentially removed. Now again fixing m , we are looking at the sum

$$(2.6) \quad \sqrt{pq} \sum_{n \in \mathbb{Z}} \sum_{r(n)} e\left(\frac{d^\circ m r + \frac{l_1 l_2 m}{d^\circ k^2} \bar{r}}{n}\right) \overline{\chi(n) \psi(n)} \sum_{\substack{d|n \\ (c_1, n/d)=1}} \sum_{\substack{c_1 \equiv 0(p) \\ c_2 = -(c_1 r + \lambda \frac{n}{d}), c_2 \equiv 0(q)}} \sum_{\substack{(\lambda, c_1 d)=1}} \chi(dc_2) \psi(dc_1) H_n\left(\frac{dc_1}{pq}, \frac{dc_2}{pq}\right),$$

which is similar to our (6.10). Here H_n is nice test function defined in terms of the test functions V, W in (6.11). Denote the Mellin transform of H_n by \widetilde{H}_n . Then we can write (2.6)

using Mobius inversion as

$$(2.7) \quad \frac{\sqrt{pq}}{(pq)^2} \sum_{n \in \mathbb{Z}} \sum_{r(n)} e\left(\frac{d^\circ mr + \frac{l_1 l_2 m}{d^\circ k^2} \bar{r}}{n}\right) \overline{\chi(n) \psi(n)} \sum_{d|n} \sum_{\substack{c_1 \equiv 0(p) \\ (c_1, n/d)=1}} \psi(dc_1) \sum_{f|c_1} \mu(f) \sum_{(\lambda, c_1 d)=1} \chi(-f \lambda q n) \times \\ \int_{(s_1)=10} \int_{(s_2)=10} \widetilde{H}_n(s_1, s_2) \left(\frac{pq}{dc_1}\right)^{s_1} \left(\frac{pq}{dqf(-c_1 r + \frac{\lambda n}{d})}\right)^{s_2} ds_1 ds_2.$$

The first step in our estimation is that we can truncate the n -sum with a negligible error (Lemma 6.7) and get the following proposition

Proposition 2.1 (Step one of estimate). *Let p, q be distinct primes and $\delta, \theta > 0$ be small but fixed. Let $H_n(x)$ be defined as in (6.11). We have*

$$(2.8) \quad \frac{\sqrt{pq}}{(pq)^2} \sum_{\substack{n \neq 0 \\ n \ll (pq)^\theta L^2}} \sum_{r(n)} e\left(\frac{d^\circ mr + \frac{l_1 l_2 m}{d^\circ k^2} \bar{r}}{n}\right) \overline{\chi(n) \psi(n)} \sum_{d|n} \sum_{\substack{c_1 \equiv 0(p) \\ (c_1, n/d)=1}} \psi(dc_1) \sum_{f|c_1} \mu(f) \sum_{(\lambda, c_1 d)=1} \chi(-f \lambda q n) \\ \int_{(s_1)=10} \int_{(s_2)=10} \widetilde{H}_n(s_1, s_2) \left(\frac{pq}{dc_1}\right)^{s_1} \left(\frac{pq}{dqf(-c_1 r + \frac{\lambda n}{d})}\right)^{s_2} ds_1 ds_2 = \\ \frac{\psi(p) \chi(-q) \sqrt{pq}}{(pq)^2} \left(\frac{1}{2\pi i}\right)^3 \int_{(s_1)=2} \int_{(s_2)=1/2+\delta} \int_{(w)=1/2} \frac{\Gamma(w) \Gamma(s_2 - w)}{\Gamma(s_2)} \frac{(pq)^{s_1+s_2}}{p^{s_1+s_2-w} q^{s_2}} \\ \frac{L(\chi, w) L(\psi, s_1 + s_2 - w)}{L(\chi \psi, s_1 + 2s_2 - w)} \left[\sum_{e=1} \frac{\mu(e) \chi(e)}{e^{w+s_1+s_2}} \sum_{d=1} \frac{1}{d^{s_1+s_2}} \sum_{a=1} \frac{\mu(a)}{a^{s_1+s_2}} \sum_{b=1} \frac{\mu^2(b) \chi(b)}{b^{s_1+2s_2}} \right. \\ \left. \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0 \\ baedn \ll (pq)^\theta L^2}} \frac{\widetilde{H}_{baedn}(s_1, s_2)}{n^w} \sum_{r(baedn)} \frac{1}{r^{s_2-w}} e\left(\frac{d^\circ mr + \frac{l_1 l_2 m}{d^\circ k^2} \bar{r}}{baedn}\right) \overline{\psi(n)} \right] dw ds_1 ds_2.$$

We point out, similar to [H], the r -sum on the left hand side of (2.8) is almost another new Kloosterman sum, the problem is that the inner c_2 depends on r . Further, if there were no arithmetic conditions on the c_1, c_2 -sums in (2.6) then an easy application of Mellin inversion or Poisson summation would suffice to understand how big the Kloosterman-Kloosterman term is with respect to p, q . However, we must face these arithmetic difficulties. Following [Y], we use the Mellin transform

$$(2.9) \quad \frac{1}{(1+x)^s} = \frac{1}{2\pi i} \int_{(w)} \frac{\Gamma(w) \Gamma(s-w)}{\Gamma(s)} x^{-w} dw$$

to analytically “separate” the λ -sum from the c_1 -sum.

Now we make a contour shift that passes a pole to give a main term and a remainder term from the right hand side of (2.8). This gives (2.8) equaling

$$(2.10) \quad \frac{\sqrt{pq}\psi(p)\chi(-q)}{(pq)^2} \left(\frac{1}{2\pi i}\right)^2 \int_{(s_1)=2} \int_{(s_2)=1/2} \frac{(pq)^{s_1+s_2}}{p^{s_1}q^{s_2}} \\ \frac{L(\chi, s_2)L(\psi, s_1)}{L(\chi\psi, s_1+s_2)} \left[\sum_{e=1} \frac{\mu(e)\chi(e)}{e^{s_1+2s_2}} \sum_{d=1} \frac{1}{d^{s_1+s_2}} \sum_{a=1} \frac{\mu(a)}{a^{s_1+s_2}} \sum_{b=1} \frac{\mu^2(b)\chi(b)}{b^{s_1+2s_2}} \right. \\ \left. \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0 \\ baedn \ll (pq)^\theta L^2}} \frac{\widetilde{H_{baedn}}(s_1, s_2)}{n^{s_2}} \sum_{r(baedn)} e\left(\frac{d^\circ mr + \frac{l_1 l_2 m}{d^\circ k^2} \bar{r}}{baedn}\right) \overline{\psi(n)} \right] ds_1 ds_2 + \{\text{Remainder term(6.26)}\}.$$

In (2.10) we clearly see Dirichlet L-functions that will aid in getting subconvexity in the Rankin-Selberg L-function.

Step two of the estimation will now bring in the m -sum from (2.5) to the main term (2.10). First we define for $\Re(s_1) = 2, \Re(s_2) = 1/2$ the function

$$(2.11) \quad T_{m,v}(c) := e\left(\frac{-mv}{c}\right) \int_0^\infty \int_0^\infty e\left(\frac{xd^{\circ 2}kv}{\sqrt{l_1 l_2} cy} + \frac{y(l_1 l_2)^{3/2}v}{d^{\circ 2} k^3 cx}\right) \\ \int_{-\infty}^\infty e\left(\frac{tck}{\sqrt{l_1 l_2} vxy}\right) F_M(t) V\left(\frac{4\pi\sqrt{t}}{x}\right) W\left(\frac{4\pi\sqrt{t}}{y}\right) x^{s_1-1} y^{s_2-1} \frac{dt}{t^{1/2}} dx dy.$$

This is virtually the function $\widetilde{H}_\mu(s_1, s_2)$ with an extra exponential sum and with a change of variables $x \rightarrow \frac{\sqrt{md^\circ}x}{\sqrt{pq}}$ and $y \rightarrow \frac{\sqrt{m\frac{l_1 l_2}{d^\circ k^2}}y}{\sqrt{pq}}$.

Proposition 2.2 (Step two of estimate). *With p, q distinct primes, let $\Delta = m^2 - \frac{4l_1 l_2}{k^2}, m \in \mathbb{Z}$ and $(\frac{\Delta}{\cdot})$ the quadratic character with modulus Δ . We have for any $\epsilon > 0$ and any $A > 0$ the asymptotic equality*

$$(2.12) \quad \frac{\sqrt{pq}\psi(p)\chi(-q)}{(pq)^2} \sum_{m \geq 1} \frac{1}{m^{1/2}} F'_{M'}\left(\frac{d^\circ m}{pq}\right) \{\text{Main term of (2.10)}\} = \\ \frac{\sqrt{pq}\psi(p)\chi(-q)}{(pq)^2} \sum_m \left(\frac{1}{2\pi i}\right)^2 \int_{(s_1)=2} \int_{(s_2)=1/2} \left(\frac{l_1 l_2}{k^2}\right)^{\frac{s_1+s_2}{2}} \int_{(w)=\epsilon} \int_{-\infty}^\infty F'_{M'}\left(\frac{d^\circ v}{pq}\right) \widetilde{T_{m,v}}(w) \frac{1}{(pq)^{\frac{s_1+s_2}{2}}} \frac{(pq)^{s_1+s_2}}{p^{s_1}q^{s_2}} \\ \frac{L(s_2, \chi)L(s_1, \psi)}{L(s_1+s_2, \chi\psi)} \frac{L(s_1+2s_2+1+w, (\frac{\Delta}{\cdot})\chi)}{L(s_1+2s_2+1+w, \chi)} \frac{L(s_2+1+w, \bar{\psi})}{L(s_2+1+w, (\frac{\Delta}{\cdot})\bar{\psi})} B(s_1+2s_2+1+w) v^{\frac{s_1+s_2-1}{2}} dv dw ds_1 ds_2 \\ + O((pq)^{-A}).$$

Here $B(s)$ is an analytic function for $\Re(s) \geq 1$ defined in (6.41).

This second estimate comes from the earlier reorganization where we replaced the sum $\overline{x}(c_1)$ with the sum $r(n)$. The replacement benefits us in that the m -sum coming from the left hand side of (2.12) is a sum of Kloosterman sums that is very similar to Rudinick's thesis [R]:

$$\sum_{m \geq 1} \frac{1}{m^{1/2}} F'_{M'}\left(\frac{d^\circ m}{pq}\right) S(d^\circ m, \frac{l_1 l_2 m}{d^\circ k^2}, baedn) A(m),$$

with $A(x)$ a nice function. This equation is analogous to our (6.31). Notice also this new Kloosterman sum has trivial nebentypus even though we initially started with forms of type $\chi(p), \psi(q)$. Applying Poisson summation modulo $baedn$ leaves us counting solutions to quadratic equations. Such a count is analogous to calculating orbital integrals in the Arthur-Selberg trace formula. The beautiful thing now, though it is not essential to the estimate, is that these solutions to quadratic equations modulo l depend only the square free part of l , and are also multiplicative in l .

We will ultimately have the contour of the s_1 -integral on the line $\Re(s_1) = 1/2$. This term is analogous to our (6.42) below. Note from this term we will expect that convexity or subconvexity estimates will come into play for Dirichlet L-functions. We also make a comment that the m -sum in this case, which is dual to the m -sum from opening the norm square of the approximate functional equation, will play the role of the sum of traces of conjugacy classes in the Selberg trace formula, see [R].

To understand the third step we define

$$A := A(x, y, d^\circ, k, l_1, l_2) := \left(\frac{x d^{\circ 2} k}{\sqrt{l_1 l_2} y} + \frac{y (l_1 l_2)^{3/2}}{d^{\circ 2} k^3 x} - m \right)$$

and

$$B := B(x, y, k, l_1, l_2) := \frac{k}{\sqrt{l_1 l_2} x y}.$$

Then we can rewrite the v -integral in (2.12) as

$$(2.13) \quad \left(\frac{pq}{d^\circ} \right)^{\frac{s_1 + s_2 + 2w + 1}{2}} \left[\int_0^\infty F_M(t) t^{\frac{s_1 + s_2 - 1}{2}} dt \right] \left[\int_{-\infty}^\infty F'_{M'}(v) v^{\frac{s_1 + s_2 + 2w - 1}{2}} dv \right] \int_0^\infty \int_0^\infty V\left(\frac{4\pi}{x}\right) W\left(\frac{4\pi}{y}\right) \\ \left(\frac{A}{B} \right)^{w/2} \left[\frac{J_w(4\pi \sqrt{\frac{A}{B}}) - J_{-w}(4\pi \sqrt{\frac{A}{B}})}{2 \sin(\pi w/2)} + \frac{J_w(4\pi \sqrt{\frac{A}{B}}) + J_{-w}(4\pi \sqrt{\frac{A}{B}})}{2 \cos(\pi w/2)} \right] x^{s_1 - 1} y^{s_2 - 1} dx dy.$$

The steps taken to rewrite the v -integral in this form follow from equations (6.43) to (6.53).

Let

$$(2.14) \quad D_{\frac{A}{B}}(s_1, s_2, w) := \int_0^\infty \int_0^\infty V\left(\frac{4\pi}{x}\right) W\left(\frac{4\pi}{y}\right) \times \\ \left(\frac{A}{B} \right)^{w/2} \left[\frac{J_w(4\pi \sqrt{\frac{A}{B}}) - J_{-w}(4\pi \sqrt{\frac{A}{B}})}{2 \sin(\pi w/2)} + \frac{J_w(4\pi \sqrt{\frac{A}{B}}) + J_{-w}(4\pi \sqrt{\frac{A}{B}})}{2 \cos(\pi w/2)} \right] x^{s_1 - 1} y^{s_2 - 1} dx dy$$

and

$$(2.15) \quad C(s_1, s_2, w) := \frac{L(s_2, \chi) L(s_1, \psi)}{L(s_1 + s_2, \chi \psi)} \frac{L(s_1 + 2s_2 + 1 + w, (\frac{\Delta}{\cdot}) \chi)}{L(s_1 + 2s_2 + 1 + w, \chi)} \frac{L(s_2 + 1 + w, \bar{\psi})}{L(s_2 + 1 + w, (\frac{\Delta}{\cdot}) \bar{\psi})} B(s_1 + 2s_2 + 1 + w).$$

Then the right hand side of step two equals

$$(2.16) \quad \frac{\psi(p)\chi(-q)}{pq} \sum_m \left(\frac{1}{2\pi i}\right)^2 \int_{(s_1)=2} \int_{(s_2)=1/2} \frac{(pq)^{s_1+s_2}}{p^{s_1}q^{s_2}} \left(\frac{l_1 l_2}{k^2}\right)^{\frac{s_1+s_2}{2}} \times \\ \int_{(w)=\epsilon} \left(\frac{pq}{d^\circ}\right)^w \left[\int_0^\infty F_M(t) t^{\frac{s_1+s_2-1}{2}} dt \right] \left[\int_{-\infty}^\infty F'_{M'}(v) v^{\frac{s_1+s_2+2w-1}{2}} dv \right] D_{\frac{A}{B}}(s_1, s_2, w) C(s_1, s_2, w) dw ds_1 ds_2 \\ + O((pq)^{-A}).$$

From here a subconvexity estimate is needed for $C(s_1, s_2, w)$ and Bessel function analysis is required for $D_{\frac{A}{B}}(s_1, s_2, w)$. We also point out the interesting fact here that the functions $F_M, F'_{M'}$ which come from the approximate functional equation are “unhinged” from the actual subconvexity calculation. This is good as one can always change the test functions involved in the approximate functional equation. We say though, that these integrals of F_M and $F'_{M'}$ (via integration by parts) are important in ensuring the convergence of the s_1, s_2 and w -integrals.

Proposition 2.3 (Step three of estimate). *For any $\epsilon > 0$ and A and B defined as above, we have*

•

$$C(1/2 + it_1, 1/2 + it_2, iy) \ll (|t_1|p)^{\theta_1} (|t_2|q)^{\theta_2} (|t_1 t_2 y|pq)^\epsilon$$

where θ_1, θ_2 are parameters $\leq 1/4$.

•

$$D_{\frac{A}{B}}(1/2 + it_1, 1/2 + it_2, i\gamma) \ll (t_1 t_2 \gamma \frac{A}{B})^\epsilon.$$

We mention now the remainder term from (2.10) can be analyzed in the same manner as in steps two and three. We unfortunately do not get such a nice product of L-functions as in step two, but analogous analysis of that of the main term gets the same bound as the main term times an arbitrarily small but fixed power of q . We write these two estimates as our final step four of the analysis of the Kloosterman-Kloosterman term.

Proposition 2.4 (Step four of estimate). *For a fixed but arbitrarily small $\delta > 0$, we have*

•

$$(2.17) \quad \frac{\sqrt{pq}\psi(p)\chi(-q)}{(pq)^2} \sum_{m \geq 1} \frac{1}{m^{1/2}} F'_{M'}\left(\frac{d^\circ m}{pq}\right) \{\text{Main term of (2.10)}\} \ll \frac{p^{\theta_1+\epsilon} q^{\theta_2+\epsilon}}{\sqrt{pq}};$$

•

$$(2.18) \quad \frac{\sqrt{pq}\psi(p)\chi(-q)}{(pq)^2} \sum_{m \geq 1} \frac{1}{m^{1/2}} F'_{M'}\left(\frac{d^\circ m}{pq}\right) \{\text{Remainder term of (2.10)}\} \ll \frac{p^{\theta_1+\epsilon} q^{\delta+\theta_2+\epsilon}}{\sqrt{pq}};$$

•

$$(2.19) \quad \{\text{Equation (2.5)}\} \ll \frac{(1+q^\delta) p^{\theta_1+\epsilon} q^{\theta_2+\epsilon}}{\sqrt{pq}}$$

where θ_1, θ_2 are parameters $\leq 1/4$.

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3. RANKIN-SELBERG L-FUNCTION

Let χ and ψ be Dirichlet characters modulo prime numbers p and q , respectively, with $(p, q) = 1$. Let $f \in H(p, \chi)$ and $g \in H(q, \psi)$ be newforms associated to GL_2 cuspidal automorphic representations. The Rankin-Selberg L-function is

$$(3.1) \quad L(s, f \times g) := L(\chi\psi, 2s) \sum_{n=1}^{\infty} \frac{a_f(n)a_g(n)}{n^s}.$$

It is known that this L-function has analytic continuation to the complex plane. Following [KMV] we can write the Rankin-Selberg L-function as a truncated Dirichlet series using an approximate functional equation. Namely, we can write

$$L(1/2, f \times g) = \sum_{n \geq 1} \frac{a_f(n)a_g(n)}{n^{1/2}} H\left(\frac{n}{pq}\right) + \epsilon(f \times g) \sum_{n \geq 1} \frac{\overline{a_f(n)a_g(n)}}{n^{1/2}} H'\left(\frac{n}{pq}\right),$$

with $|\epsilon(f \times g)| = 1$ and defined in [KMV]. The function H (resp. H') satisfies $H(y) \ll y^{-A}$, for all $A > 0$. As well for y small H (resp. H') satisfies

$$H(y) \ll (pq)^\epsilon |\log y|.$$

We will also note that in estimating from line to line $\{expression\}^\epsilon$ will note bounding by an arbitrarily small positive power, and is not necessarily the same ϵ line to line.

4. KUZNETSOV TRACE FORMULA

We start by defining the Kuznetsov trace formula. We refer to [Iw] book on it's derivation. Let $S(\Gamma_0(p), \chi)$ be the space of holomorphic cusp forms of weight k for the group $\Gamma_0(p)$. For each form $\phi \in S_k(\Gamma_0(p), \chi)$, let $c_n(\phi)$ be the n -th Fourier coefficient, then define

$$a_n(\phi) := \sqrt{\frac{\pi^{-k}\Gamma(k)}{(4n)^{k-1}}} c_n(\phi).$$

Likewise, for Maass cusp forms we define

$$a_n(\phi) := \left(\frac{4\pi|n|}{\cosh(\pi s)}\right)^{1/2} \rho(n),$$

where ϕ has L^2 norm one and eigenvalue $1/4 + s^2$ with Fourier expansion

$$\phi(z) = \sum_{n \neq 0} \rho(n) W_s(nz).$$

Here $W_s(nz) = 2\sqrt{y} K_{s-1/2}(ny) e(x)$. The continuous spectrum coefficients are defined as

$$\eta(l, 1/2 + it) := 2\pi^{1+it} \cosh(\pi t)^{-1/2} \frac{\tau_{it}(n)}{\Gamma(1/2 + it) \zeta(1 + 2it)},$$

where $\tau_{it}(n) = \sum_{ab=n} (a/b)^{it}$.

Let $V \in C_0^\infty(\mathbb{R}^+)$, then the Kuznetsov formula states

$$(4.1) \quad \sum_{\phi} h(V, \lambda_{\phi}) a_n(\phi) \overline{a_l(\phi)} + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(V, t) \eta(n, 1/2 + it) \overline{\eta(l, 1/2 + it)} dt$$

$$= \frac{\delta_{n,l}}{\pi} \left[\int_{-\infty}^{\infty} h(V, t) \tanh(\pi t) t dt + \sum_{k > 0, k \text{ even}} (k-1) h(V, k) \right] + \sum_{c \equiv 0(p)}^{\infty} \frac{1}{c} S_{\chi}(l, n, c) V(4\pi\sqrt{ln}/c),$$

where the sum ϕ is over an orthonormal basis for $S_k(\Gamma_0)$, $k \in 2\mathbb{Z}$ and Maass forms w.r.t. the Petersson inner product, and $V \in C_0^\infty(\mathbb{R} - \{0\})$. The Kloosterman sum is

$$S_\chi(l, n, c) = \sum_{x(c)^*} \chi(x) e\left(\frac{lx + n\bar{x}}{c}\right),$$

as well

$$(4.2) \quad h(V, \lambda) := \begin{cases} i^k \int_0^\infty V(x) J_{\lambda-1}(x) x^{-1} dx & \text{if } \lambda \in 2\mathbb{Z}; \\ \int_0^\infty V(x) B_{2i\lambda}(x) x^{-1} dx & \text{if } \lambda \in \mathbb{R} - 2\mathbb{Z}. \end{cases}$$

Here, $B_{2it}(x) = (2 \sin(\pi it))^{-1} (J_{-2it}(x) - J_{2it}(x))$, where $J_\mu(x)$ is the standard J -Bessel function of index μ (See [IK] and [Wat]).

The delta (main) term of the geometric side of the trace formula has associated to it

$$\delta_{a,b} := \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases}$$

5. INCLUDING AN AMPLIFIER

We build in an amplifier, and thus our starting point is

$$(5.1) \quad \sum_{f \in H(p, \chi)} h(V, t_f) \left| \sum_{l \leq L} x_l \lambda_f(l) \right|^2 \sum_{h \in H(q, \psi)} h(W, t_h) |L(1/2, f \times h)|^2,$$

where x_l are arbitrary complex coefficients. Specifically, $|L(1/2, f \times h)|^2$ from the approximate functional equation above is equal to four sums given by

$$(5.2) \quad \sum_{f \in H(p, \chi), h \in H(q, \psi)} h(V, t_f) h(W, t_h) \sum_{m, n \geq 1} \frac{1}{(mn)^{1/2}} H\left(\frac{n}{pq}\right) H'\left(\frac{m}{pq}\right) a_f(n) a_f(m) a_h(n) a_h(m).$$

up to a root number $\epsilon(f \times h)$. We now introduce a partition of unity for the n - and m -sums similar to [KMV]. Let $\eta(x)$ be a smooth function which is zero for $x \leq 1/2$ and one for $x \geq 1$ such that

$$\eta(x) = \sum_{M \geq 1} \eta_M(x)$$

with η_M compactly supported in $[M/2, 2M]$ such that $x^i \eta_M(x)^{(i)} \ll_i 1$ for any $i \geq 0$. The number of partitions M less than X is $O(\log X)$. We use such a partition on the two sums n, m , and define

$$F(x) := H(x) \eta(pqx) = \sum_{M \geq 1} H(x) \eta_M(pqx) := \sum_{M \geq 1} F_M(x).$$

Following the analysis of [KMV] we also have the bound for all $i, A \geq 0$

$$(5.3) \quad x^i \frac{\partial^i}{\partial^i x} F_M(x) \ll_{A,i} \frac{(pq)^\epsilon}{M^{1/2}} \left(\frac{pq}{x}\right)^A$$

for any $\epsilon > 0$. The notation for the analogous partition on the m -sum and H' will be $\eta_{M'}, F_{M'}$. Also note by the decay in H (resp. H') in the definition of the approximate functional equation we have $M \leq (pq)^{1+\epsilon}$.

So we will study the sum

$$(5.4) \quad \sum_{M, M' \geq 1} \sum_{f \in H(p, \chi), h \in H(q, \psi)} h(V, t_f) h(W, t_h) \left| \sum_{l \leq L} x_l \lambda_f(l) \right|^2 \sum_{m, n \geq 1} \frac{1}{(mn)^{1/2}} F_M\left(\frac{n}{pq}\right) F'_{M'}\left(\frac{m}{pq}\right) \times \\ a_f(n) a_f(m) a_h(n) a_h(m).$$

Now opening the square and using Hecke properties, we get

$$\left| \sum_{l \leq L} x_l \lambda_f(l) \right|^2 = \sum_{l_1, l_2 \leq L} x_{l_1} x_{l_2} \sum_{k | (l_1, l_2)} \lambda_f\left(\frac{l_1 l_2}{k^2}\right).$$

Incorporating the above equation into (5.4) for a fixed M, M' from our partition of unity we have

$$(5.5) \quad \sum_{l_1, l_2 \leq L} x_{l_1} x_{l_2} \sum_{k | (l_1, l_2)} \sum_{d^\circ | \frac{l_1 l_2}{k^2}} \frac{1}{d^{\circ 1/2}} \sum_{m \geq 1} \frac{1}{m^{1/2}} F'_{M'}\left(\frac{d^\circ m}{pq}\right) \\ \left[\sum_{n \geq 1} \frac{1}{n^{1/2}} F_M\left(\frac{n}{pq}\right) \sum_{f \in H(p, \chi), h \in H(q, \psi)} h(V, t_f) h(W, t_h) a_f(n) a_h(d^\circ m) a_h(n) a_f\left(\frac{l_1 l_2}{d^\circ k^2} m\right) \right].$$

Let us define

$$D_{n, l}(V) := \frac{\delta_{n, l}}{\pi} \left[\int_{-\infty}^{\infty} h(V, t) \tanh(\pi t) t dt + \sum_{k > 0, k \text{ even}} (k - 1) h(V, k) \right].$$

Then performing Kuznetsov twice and Poisson summation on the n -sum in the {Kloosterman-Kloosterman term} we have, as in section 7 of [H],

$$\begin{aligned}
(5.6) \quad & \sum_{l_1, l_2 \leq L} x_{l_1} x_{l_2} \sum_{q|(l_1, l_2)} \sum_{d^\circ | \frac{l_1 l_2}{k^2}} \frac{1}{d^{\circ 1/2}} \sum_{m \geq 1} \frac{1}{m^{1/2}} F'_{M'}\left(\frac{d^\circ m}{pq}\right) \times \\
& \left\{ \sum_{n \geq 1} \frac{1}{n^{1/2}} F_M\left(\frac{n}{pq}\right) D_{n, \frac{l_1 l_2 m}{d^\circ k^2}}(V) D_{n, d^\circ m}(W) \{\text{delta-delta term}\} + \right. \\
& \sum_{n \geq 1} \frac{1}{n^{1/2}} F_M\left(\frac{n}{pq}\right) D_{n, \frac{l_1 l_2 m}{d^\circ k^2}}(V) \left(\sum_{c_2 \equiv 0(q)} \frac{S_\psi(n, d^\circ m, c_2)}{c_2} W\left(\frac{4\pi \sqrt{nd^\circ m}}{c_2}\right) \right) \{\text{delta-Kloosterman term}\} + \\
& \sum_{n \geq 1} \frac{1}{n^{1/2}} F_M\left(\frac{n}{pq}\right) D_{n, d^\circ m}(W) \left(\sum_{c_1 \equiv 0(p)} \frac{S_\chi(n, \frac{l_1 l_2 m}{d^\circ k^2}, c_1)}{c_1} V\left(\frac{4\pi \sqrt{\frac{nl_1 l_2 m}{d^\circ k^2}}}{c_1}\right) \right) \{\text{Kloosterman-delta term}\} + \\
& \sqrt{pq} \sum_{\substack{c_1 \equiv 0(p) \\ c_2 \equiv 0(q)}} \frac{1}{c_1 c_2} \sum_{x(c_1)^*} \sum_{y(c_2)^*} \chi(\bar{x}) \psi(\bar{y}) e\left(\frac{\bar{x} d^\circ m}{c_1} + \frac{\bar{y} \frac{l_1 l_2 m}{d^\circ k^2}}{c_2}\right) \times \\
& \left\{ \sum_k \int_{-\infty}^{\infty} e\left(pqt \left(\frac{c_2 x + c_1 y - k c_1 c_2}{c_1 c_2}\right)\right) \frac{F_M(t)}{t^{1/2}} V\left(\frac{4\pi \sqrt{tpq d^\circ m}}{c_1}\right) W\left(\frac{4\pi \sqrt{tpq \frac{l_1 l_2 m}{d^\circ k^2}}}{c_2}\right) dt \right\} \\
& \left. \{\text{Kloosterman-Kloosterman term}\} \right\}.
\end{aligned}$$

We will deal with the hardest case first (Kloosterman-Kloosterman term) in Section 6 and Section 7, then in Section 8 we deal with the delta-delta term. In Section 9 following that, we deal with both the delta-Kloosterman term and Kloosterman-delta term.

6. KLOOSTERMAN-KLOOSTERMAN TERM

We let $n = c_2 x + c_1 y - k c_1 c_2$, and reparametrize the Kloosterman-Kloosterman term of (5.6) analogous to the equations up to (7.6) of [H], quoting the propositions we need from that paper. The only difference between lemmas we need here and that of [H] is that, $c_1 \equiv 0(p)$ and $c_2 \equiv 0(q)$ as there we have only full level. This does not affect the proofs of this reparametrization and so we state the propositions we need from [H].

Definition 6.1. Let $\overline{X}(c_1, c_2, n)$ denote the equivalence classes of pairs (x, y) with $x, y \in \mathbf{Z}$ such that $(x, c_1) = 1$, $(y, c_2) = 1$, and

$$c_2 x + c_1 y = n.$$

Here we say that (x, y) is equivalent to (x', y') if $x \equiv x' \pmod{c_1}$ and $y \equiv y' \pmod{c_2}$. Let $X(c_1, c_2, n)$ be a set of representatives for the classes in $\overline{X}(c_1, c_2, n)$.

Proposition 6.2. Let $(x, y) \in X(c_1, c_2, 0)$, then $x = -y, c_1 = c_2$. This implies $c_1 = c_2 \equiv 0(pq)$.

Proposition 6.3. *Let $(x, y) \in X(c_1, c_2, n)$ and $\bar{x} \in \mathbf{Z}$ be an inverse of x modulo c_1 and $\bar{y} \in \mathbf{Z}$ be an inverse of y modulo c_2 . Then there exists a pair (r_1, r_2) such that $r_1 r_2 \equiv 1 \pmod{n}$ and*

$$(6.1) \quad \bar{x} = \frac{c_2 + c_1 r_1}{n}, \quad \bar{y} = \frac{c_1 + c_2 r_2}{n}$$

The pair (r_1, r_2) is uniquely determined modulo n by the equivalence class of the pair (x, y) , and the map from $X(c_1, c_2, n)$ to the set of pairs (r_1, r_2) modulo n is injective.

Definition 6.4. *Let c_1, c_2 be positive integers. Set $d = (c_1, c_2)$. Assume that $d|n$. Let $Y(c_1, c_2, n)$ be the set of classes $r \in (\mathbf{Z}/n)^*$ such that*

- (a') $(c_1/d)r + (c_2/d) \equiv 0 \pmod{\frac{n}{d}}$
- (b') $(c_1/d)r + (c_2/d) \not\equiv 0 \pmod{\frac{n}{d'}}$ if $d'|d$ and $d' < d$.

Proposition 6.5. *The map $i : (x, y) \rightarrow r_1$ defines a bijection between $X(c_1, c_2, n)$ and $Y(c_1, c_2, n)$.*

For proofs of Propositions 6.2, 6.3 and 6.5, see [H].

Now we break the Kloosterman-Kloosterman term into 2 cases: $n = 0$ and $n \neq 0$.

6.1. [Case $n = 0$]. This is by far the easier of the two cases. Using Proposition 6.2 above for $n = 0$, we must have $c_1 = c_2 \equiv 0(pq)$ and $\bar{x} = -\bar{y}$.

$$(6.2) \quad \sqrt{pq} \sum_{m \geq 1} \frac{1}{m^{1/2}} F'_{M'}\left(\frac{d^\circ m}{pq}\right) \sum_{c_1 \equiv 0(qp)} \frac{1}{c_1^2} \sum_{x(c_1)^*} \chi(\bar{x}) \psi(-\bar{x}) e\left(\frac{\bar{x}m(d^\circ - \frac{l_1 l_2}{d^\circ k^2})}{c_1}\right) \times$$

$$\int_{-\infty}^{\infty} \frac{F_M(t)}{t^{1/2}} V\left(\frac{4\pi\sqrt{tpqd^\circ m}}{c_1}\right) W\left(\frac{4\pi\sqrt{tpq\frac{l_1 l_2 m}{d^\circ k^2}}}{c_1}\right) dt.$$

Note if $d^\circ - \frac{l_1 l_2}{d^\circ k^2} = 0$, then character sum is zero as $\psi \neq \chi$. So we assume for now on $d^\circ - \frac{l_1 l_2}{d^\circ k^2} \neq 0$.

Now the x -sum above is a Gauss sum with primitive character $\chi\psi$ of conductor pq . In order for this sum to be non-zero we must have

$$e\left(\frac{\bar{x}m(d^\circ - \frac{l_1 l_2}{d^\circ k^2})}{c_1}\right) = e\left(\frac{\bar{x}m'a}{pq}\right),$$

where $m'|m, a|(d^\circ - \frac{l_1 l_2}{d^\circ k^2})$, and $(m'a, pq) = 1$. Further we must have $c_1 = pqr$, where $(r, pq) = 1$ otherwise the whole term (6.2) is zero as the support of V, W , and g implies $m \sim pq$ and so r must be independent of p, q .

Then by Chinese remainder theorem the Gauss sum equals

$$(6.3) \quad \sum_{x(c_1)^*} \chi(\bar{x}) \psi(-\bar{x}) e\left(\frac{\bar{x}m(d^\circ - \frac{l_1 l_2}{d^\circ k^2})}{c_1}\right) = \left[\sum_{x(p)^*} \chi(\bar{x}) e\left(\frac{\overline{qr} \bar{x} m(d^\circ - \frac{l_1 l_2}{d^\circ k^2})}{p}\right) \right] \times$$

$$\left[\sum_{y(q)^*} \psi(\bar{y}) e\left(\frac{\overline{pr} \bar{y} m(d^\circ - \frac{l_1 l_2}{d^\circ k^2})}{q}\right) \right] \left[\sum_{z(r)^*} e\left(\frac{\overline{pq} \bar{z} m(d^\circ - \frac{l_1 l_2}{d^\circ k^2})}{r}\right) \right].$$

Now to get a non-zero contribution for (6.2),

$$(m(d^\circ - \frac{l_1 l_2}{d^\circ k^2}), pq) = 1.$$

With an easy change of variables, (6.3) equals

$$\overline{\chi(qr)\psi(pr)}\overline{\chi(m(d^\circ - \frac{l_1 l_2}{d^\circ k^2}))}\overline{\psi(m(d^\circ - \frac{l_1 l_2}{d^\circ k^2}))}\tau(\chi)\tau(\psi)f_{\frac{c_1}{pq}}(m(d^\circ - \frac{l_1 l_2}{d^\circ k^2})),$$

where

$$f_n(m) = \sum_{b|(m,n)} \mu\left(\frac{n}{b}\right)b,$$

$\tau(\chi), \tau(\psi)$ are Gauss sums, and $r = \frac{c_1}{pq}$.

Let

$$(6.4) \quad Z(x) := \frac{1}{x} \int_{-\infty}^{\infty} \frac{F_M(t)}{t^{1/2}} V\left(\frac{4\pi\sqrt{td^\circ m}}{x}\right) W\left(\frac{4\pi\sqrt{\frac{l_1 l_2 m}{d^\circ k^2}} t}{x}\right) dt.$$

Denoting the Mellin transform of $Z(x)$ as $\tilde{Z}(s)$, we use Mellin inversion to write the c_1 -sum in (6.2) as

$$(6.5) \quad \frac{1}{\sqrt{pq}} \sum_{c_1 \equiv 0(pq)} \frac{f_{\frac{c_1}{pq}}(m(d^\circ - \frac{l_1 l_2}{d^\circ k^2}))}{c_1} Z\left(\frac{c_1}{\sqrt{pq}}\right) = \left\{ \frac{1}{2\pi i} \int_{(\sigma)} \tilde{Z}(s) L(s) (\sqrt{pq})^s ds \right\}$$

for $\sigma > 0$ large, and

where

$$L(s) := \sum_{c_1 \equiv 0(pq)} \frac{f_{\frac{c_1}{pq}}(m(d^\circ - \frac{l_1 l_2}{d^\circ k^2}))}{c_1^{1+s}} = \frac{1}{(pq)^{1+s}} \left[\sum_{b|m(d^\circ - \frac{l_1 l_2}{d^\circ k^2})} \frac{1}{b^s} \right] \sum_{c_1=1}^{\infty} \frac{\mu(c_1)}{c_1^{1+s}}.$$

The second equality follows from inverting the c_1 - and b -sum.

In (6.5) we can shift the contour again to $\Re(s) = 1$ bounding the b -sum by $(m(d^\circ - \frac{l_1 l_2}{d^\circ k^2}))^\epsilon$. We have left to estimate

$$(6.6) \quad \frac{\tau(\chi)\tau(\psi)}{(pq)^{3/2}} \sum_{m \geq 1} \frac{\chi(m)\psi(m)m^\epsilon}{m^{1/2}} F'_{M'}\left(\frac{d^\circ m}{pq}\right) \int_{(1)} \left[\int_{-\infty}^{\infty} \frac{F_M(t)}{t^{1/2}} V\left(\frac{4\pi\sqrt{td^\circ m}}{x}\right) W\left(\frac{4\pi\sqrt{\frac{l_1 l_2 m}{d^\circ k^2}} t}{x}\right) dt x^{s-2} dx \right] ds.$$

Let

$$Z'(z) := F'_{M'}\left(\frac{d^\circ z}{pq}\right) \int_{(1)} \left[\int_{-\infty}^{\infty} \frac{F_M(t)}{t^{1/2}} V\left(\frac{4\pi\sqrt{td^\circ z}}{x}\right) W\left(\frac{4\pi\sqrt{\frac{tl_1 l_2 z}{d^\circ k^2}}}{x}\right) dt x^{s-2} dx \right] ds.$$

Then (6.6) equals

$$\frac{\tau(\chi)\tau(\psi)}{(pq)^{3/2}} \frac{1}{2\pi i} \int_{(\sigma_1)} \tilde{Z}'(w) L(1/2 + w - \epsilon, \chi\psi) (pq)^w dw.$$

We can shift this contour to $\Re(w) = \epsilon$ where $\epsilon > 0$ without hitting a pole. Then we note

$$\tilde{Z}'(w) = \int_0^\infty F'_{M'}\left(\frac{d^\circ v}{pq}\right) \left[\int_{(1)} \int_{-\infty}^{\infty} \frac{F_M(t)}{t^{1/2}} V\left(\frac{4\pi\sqrt{td^\circ v}}{x}\right) W\left(\frac{4\pi\sqrt{\frac{tl_1 l_2 v}{d^\circ k^2}}}{x}\right) dt x^{s-2} dx \right] ds v^{w-1} dv \ll (pq)^\epsilon,$$

and also by the support of the functions V, W , we get $d^\circ k \sim \sqrt{l_1 l_2}$.

The convexity bound in the level aspect for a Dirichlet L-function, with the level q , is

$$L(1/2 + it, \chi) \ll q^{1/4+\epsilon}.$$

A subconvexity bound is

$$L(1/2 + it, \chi) \ll q^{\theta_1+\epsilon},$$

where $\theta_1 < 1/4$. Not to discriminate using a convexity or subconvexity bound in this paper, we call it a (sub)convexity bound and note (6.6) is bounded by

$$(pq)^{\theta_1-1+\epsilon}.$$

Therefore using the fact that $d^\circ q \sim \sqrt{l_1 l_2}$, with the sum over l_1, l_2 , we have

$$(6.7) \quad (pq)^{\theta_1-1+\epsilon} \sum_{l_1, l_2 \leq L} \frac{x_{l_1} x_{l_2}}{(l_1 l_2)^{1/4}} \sum_{q|(l_1, l_2)} \sqrt{q} \ll (pq)^{\theta_1-1+\epsilon} L^\epsilon \|x\|_2^2.$$

6.2. [Case $n \neq 0$].

6.2.1. *Explanation of this case.* Let us summarize what happens in this case, as it is the most complicated case in the paper. After using the bijection of [H], we have reparametrized the c_1, c_2 , and k -sums in (5.6) into a new n -sum. This puts arithmetic conditions on the c_1, c_2 -sums that we remove by standard techniques (Möbius inversion, inverting sums). This leaves us with the result of Proposition 6.8. From there we apply Poisson summation in the m -sum in Section 6.3 coming from (6.9). This lets us exchange Kloosterman sums for counting solutions to quadratic equations as in the thesis of Rudnick [R]. This same Poisson summation in [R] gives entry to the geometric side of the Selberg trace formula. Now we can write the leftover sums coming from the previous Möbius inversion and Poisson summation in terms of a product of various L-functions. The sizes of these L-functions will directly control how large the final estimation is for (5.1). The expression in terms of L-functions is seen in (6.42). From there we require bounding the archimedean integrals that arise in the calculation. After several changes of variables it is evident that the important archimedean estimation comes from an unexpected Bessel function that arises. Estimating these integrals, including the Bessel function, will only depend on the amplifier parameter L . We see that specific dependence only on the amplifier in the analysis of Proposition 6.9. Then we include the terms from the amplifier itself and do a final estimation.

6.2.2. *Analysis of $n \neq 0$ case.* Using the bijection we get $\bar{x} = \frac{c_1 r + c_2}{n}(c_1)$, and $\chi(\bar{x}) = \chi(c_2)\chi(\bar{n})$ with certain conditions on the c_1, c_2 -sums which are explicated in section 8 of [H].

Let

$$(6.8) \quad I(n, x, y) := \int_{-\infty}^{\infty} e\left(\frac{tn}{xy}\right) F_M(t) V\left(\frac{4\pi \sqrt{\frac{td^\circ m}{pq}}}{x}\right) W\left(\frac{4\pi \sqrt{\frac{t_1 l_1 l_2 m}{d^\circ k^2 pq}}}{y}\right) \frac{dt}{t^{1/2}},$$

then following [H] exactly we have

$$(6.9) \quad \{\text{Kloo.-Kloo. term}\} = \sqrt{pq} \sum_{n \in \mathbf{Z}} \sum_{m \geq 1} \frac{1}{m^{1/2}} F'_{M'}\left(\frac{d^\circ m}{pq}\right) \sum_{\substack{c_1 \equiv 0(p) \\ c_2 \equiv 0(q)}} \frac{1}{c_1 c_2} \\ \sum_{(x, y) \in X(c_1, c_2, n)} \chi(\bar{x}) \psi(\bar{y}) e\left(\frac{\bar{x}l}{c_1} + \frac{\bar{y}l'}{c_2}\right) I\left(n, \frac{c_1}{pq}, \frac{c_2}{pq}\right)$$

Following the argument of section 8 of [H] closely, the {Kloosterman-Kloosterman term} (without the $n = 0$ term from last section) equals

$$(6.10) \quad \frac{\sqrt{pq}}{(pq)^2} \sum_{m \geq 1} \frac{1}{m^{1/2}} F'_{M'}\left(\frac{d^\circ m}{pq}\right) \sum_{n \neq 0 \in \mathbb{Z}} \sum_{r(n)} e\left(\frac{d^\circ m r + \frac{l_1 l_2 m}{d^\circ k^2} \bar{r}}{n}\right) \overline{\chi(n) \psi(n)} \sum_{d|n} \\ \sum_{\substack{c_1 \equiv 0(p) \\ (c_1, n/d)=1}} \sum_{\substack{c_2 = -(c_1 r + \lambda \frac{n}{d}), c_2 \equiv 0(q) \\ (\lambda, c_1 d)=1}} \chi(dc_2) \psi(dc_1) H_n\left(\frac{dc_1}{pq}, \frac{dc_2}{pq}\right),$$

where

$$(6.11) \quad H_n(x, y) := \frac{1}{xy} e\left(\frac{x dm}{ny} + \frac{y \frac{l_1 l_2 m}{d k^2}}{nx}\right) I(n, x, y).$$

This is equivalent to fixing n, m , and d from (6.10) and looking at

$$(6.12) \quad \sum_{\substack{c_1 \equiv 0(p) \\ (c_1, n/d)=1}} \psi(dc_1) \sum_{\substack{c_2 = -c_1 r + \lambda \frac{n}{d}, c_2 \equiv 0(q) \\ (c_2, c_1)=1 \\ (\lambda, d)=1}} \chi(dc_2) H_n\left(\frac{dc_1}{pq}, \frac{dc_2}{pq}\right).$$

We apply Mobius inversion on the condition $(c_1, c_2) = 1$ to get

$$(6.13) \quad \sum_{\substack{c_1 \equiv 0(p) \\ (c_1, n/d)=1}} \psi(dc_1) \sum_{f|c_1} \mu(f) \sum_{\substack{c_2 = -f(c_1 r + \lambda \frac{n}{d}), f c_2 \equiv 0(q) \\ (\lambda, d)=1}} \chi(df c_2) H_n\left(\frac{dc_1}{pq}, \frac{df c_2}{pq}\right).$$

Using the fact that χ is a character modulo p and $p|c_1$, χ will only depend on $-f\lambda n$, leaving

$$(6.14) \quad \sum_{\substack{c_1 \equiv 0(p) \\ (c_1, n/d)=1}} \psi(dc_1) \sum_{f|c_1} \mu(f) \sum_{(\lambda, d)=1} \chi(-f\lambda q n) H_n\left(\frac{dc_1}{pq}, \frac{dqf(-c_1 r + \frac{\lambda n}{d})}{pq}\right).$$

We will be interchanging many sums and integrals so we require an estimate that allows to interchange the n -sum.

Definition 6.6. A function F of p, q will be called negligible if for any $A > 0$ we have

$$F(p, q) \ll (pq)^{-A}.$$

Lemma 6.7. The n -sum can be limited to length $O((pq)^\theta L^2)$ for a small but fixed $\theta > 0$, while the complementary sum is negligible.

Proof. By integration by parts on the integral $I(n, x, y)$ we get

$$(6.15) \quad \sum_{i+j+k=1} \frac{xy}{2\pi i X n} \int_{-\infty}^{\infty} e\left(\frac{tn}{xy}\right) F_M^{(k)}(t) \left(\left(\frac{2\pi \sqrt{\frac{d^\circ m}{pqt}}}{x} \right)^i V^{(i)}\left(\frac{4\pi \sqrt{\frac{td^\circ m}{pq}}}{x}\right) \right) \times \\ \left(\frac{2\pi \sqrt{\frac{t_1 l_1 l_2 m}{d^\circ k^2 pqt}}}{y} \right)^j W^j\left(\frac{4\pi \sqrt{t \frac{l_1 l_2 m}{pq d^\circ k^2}}}{y}\right) dt.$$

Notice by the compact support of V, W , and $F'_{M'}$ we have $d^\circ m \sim pq, t = O(1)$, so x is bounded and $y \sim \frac{l_1 l_2}{d^\circ k^2}$. So we have $\frac{xy}{n} \ll \frac{L^2}{n}$. The functions V, W, F_M are smooth and independent of

p, q except for the function $F_M(x)$. One gets from (5.3) the estimate $F_M^{(i)}(x) \ll (pq)^\epsilon$ for any $\epsilon > 0$. Assuming $n \gg (pq)^\theta L^2$ then by the above analysis (6.15) is bounded by $O((pq)^{\epsilon-\theta})$. We choose $\epsilon < \theta$ so that the integral decays in p and q . Then integration by parts Q times gives the bound for (6.15) of $O((pq)^{-Q\theta+\epsilon})$. For Q large enough and the n -sum in this range, (6.8) is negligible. \square

We now can limit the n -sum to a length $O((pq)^\theta L^2)$ for a fixed but small $\theta > 0$. Using the lemma and including the n - and d -sums into (6.14) we now prove

Proposition 6.8. *Let $\delta, \theta > 0$ be small but fixed. We have*

$$(6.16) \quad \frac{\sqrt{pq}}{(pq)^2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0 \\ n \ll (pq)^\theta L^2}} \sum_{r(n)} e\left(\frac{d^\circ mr + \frac{l_1 l_2 m}{d^\circ k^2} \bar{r}}{n}\right) \overline{\chi(n) \psi(n)} \sum_{d|n} \sum_{\substack{c_1 \equiv 0(p) \\ (c_1, n/d)=1}} \psi(dc_1) \sum_{f|c_1} \mu(f) \sum_{(\lambda, d)=1} \chi(-f \lambda q n) \\ \int_{(s_1)=10} \int_{(s_2)=10} \widetilde{H}_n(s_1, s_2) \left(\frac{pq}{dc_1}\right)^{s_1} \left(\frac{pq}{dqf(-c_1 r + \frac{\lambda n}{d})}\right)^{s_2} ds_1 ds_2 = \\ \frac{\psi(p) \chi(-q) \sqrt{pq}}{(pq)^2} \left(\frac{1}{2\pi i}\right)^3 \int_{(s_1)=2} \int_{(s_2)=1/2+\delta} \int_{(w)=1/2} \frac{\Gamma(w) \Gamma(s_2 - w)}{\Gamma(s_2)} \frac{(pq)^{s_1+s_2}}{p^{s_1+s_2-w} q^{s_2}} \\ \frac{L(\chi, w) L(\psi, s_1 + s_2 - w)}{L(\chi \psi, s_1 + 2s_2 - w)} \left[\sum_{e=1} \frac{\mu(e) \chi(e)}{e^{w+s_1+s_2}} \sum_{d=1} \frac{1}{d^{s_1+s_2}} \sum_{a=1} \frac{\mu(a)}{a^{s_1+s_2}} \sum_{b=1} \frac{\mu^2(b) \chi(b)}{b^{s_1+2s_2}} \right. \\ \left. \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0 \\ baedn \ll (pq)^\theta L^2}} \frac{\widetilde{H_{baedn}}(s_1, s_2)}{n^w} \sum_{r(baedn)} \frac{1}{r^{s_2-w}} e\left(\frac{d^\circ mr + \frac{l_1 l_2 m}{d^\circ k^2} \bar{r}}{baedn}\right) \overline{\psi(n)} \right] dw ds_1 ds_2.$$

Proof. Incorporating the n and d sum back into (6.14), and applying Mellin inversion to the H_n function we have

$$(6.17) \quad \frac{\sqrt{pq}}{(pq)^2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0 \\ n \ll (pq)^\theta L^2}} \sum_{r(n)} e\left(\frac{d^\circ mr + \frac{l_1 l_2 m}{d^\circ k^2} \bar{r}}{n}\right) \overline{\chi(n) \psi(n)} \sum_{d|n} \sum_{\substack{c_1 \equiv 0(p) \\ (c_1, n/d)=1}} \psi(dc_1) \sum_{f|c_1} \mu(f) \sum_{(\lambda, d)=1} \chi(-f \lambda q n) \\ \int_{(s_1)} \int_{(s_2)} \widetilde{H}_n(s_1, s_2) \left(\frac{pq}{dc_1}\right)^{s_1} \left(\frac{pq}{dqf(-c_1 r + \frac{\lambda n}{d})}\right)^{s_2} ds_1 ds_2.$$

We need to arithmetically separate the c_1 - and λ -sum. Following [Y], we use the Mellin transform

$$(6.18) \quad \frac{1}{(1+x)^s} = \frac{1}{2\pi i} \int_{(w)} \frac{\Gamma(w) \Gamma(s-w)}{\Gamma(s)} x^{-w} dw$$

with $\Re(w) \leq \Re(s)$, to detect the condition

$$\left(\frac{1}{-c_1 r dqf(1 + \frac{\lambda n}{-c_1 r d})} \right)^{s_2}.$$

So the integral can be rewritten as

$$\left(\frac{1}{2\pi i}\right)^3 \int_{(s_1)} \int_{(s_2)} \int_{(w)} \widetilde{H}_n(s_1, s_2) \frac{\Gamma(w)\Gamma(s_2-w)}{\Gamma(s_2)} \left(\frac{pq}{dc_1}\right)^{s_1} \left(\frac{pq}{-c_1rdqf}\right)^{s_2} \left(\frac{-c_1rd}{\lambda n}\right)^w dw ds_1 ds_2,$$

with $\Re(s_1) = 2$, $\Re(s_2) = 1/2 + \delta$, and $\Re(w) = 1/2$ with $\delta > 0$ small.

As the n, c_1, λ -sums are bounded we can interchange them with the integrals. The next goal is to then detect the many relatively prime conditions with Mobius inversion and rewrite the sums in terms of Dirichlet L-functions. We do this in several steps for clarity. Fix n, d , and r for now. Apply Mobius inversion to the condition $(\lambda, d) = 1$ to get the λ -sum equal to

$$\sum_{e|d} \frac{\mu(e)\chi(e)}{e^w} \sum_{\lambda=1} \frac{\chi(\lambda)}{\lambda^w}.$$

Let us collect terms now from (6.17),

$$(6.19) \quad \frac{\sqrt{pq}}{(pq)^2} \chi(-q) \left(\frac{1}{2\pi i}\right)^3 \int_{(s_1)} \int_{(s_2)} \int_{(w)} \frac{\Gamma(w)\Gamma(s_2-w)}{\Gamma(s_2)} \frac{(pq)^{s_1+s_2}}{q^{s_2}} \\ \left[L(\chi, w) \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0 \\ n \ll (pq)^\theta L^2}} \frac{\widetilde{H}_n(s_1, s_2)}{n^w} \sum_{r(n)} \frac{1}{r^{s_2-w}} e\left(\frac{d^\circ mr + \frac{l_1 l_2 m}{d^\circ k^2} \bar{r}}{n}\right) \overline{\psi(n)} \sum_{d|n} \frac{\psi(d)}{d^{s_1+s_2-w}} \sum_{e|d} \frac{\mu(e)\chi(e)}{e^w} \right. \\ \left. \sum_{\substack{c_1 \equiv 0(p) \\ (c_1, n/d)=1}} \frac{\psi(c_1)}{c_1^{s_1+s_2-w}} \sum_{f|c_1} \frac{\mu(f)\chi(f)}{f^{s_2}} \right] dw ds_1 ds_2.$$

Write $c_1 = pc'_1$, renaming it c_1 and inverting the f -sum with the c_1 -sum gives,

$$(6.20) \quad \frac{\sqrt{pq}\psi(p)\chi(-q)}{(pq)^2} \left(\frac{1}{2\pi i}\right)^3 \int_{(s_1)} \int_{(s_2)} \int_{(w)} \frac{\Gamma(w)\Gamma(s_2-w)}{\Gamma(s_2)} \frac{(pq)^{s_1+s_2}}{p^{s_1+s_2-w} q^{s_2}} \\ \left[L(\chi, w) \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0 \\ n \ll (pq)^\theta L^2}} \frac{\widetilde{H}_n(s_1, s_2)}{n^w} \sum_{r(n)} \frac{1}{r^{s_2-w}} e\left(\frac{d^\circ mr + \frac{l_1 l_2 m}{d^\circ k^2} \bar{r}}{n}\right) \overline{\psi(n)} \sum_{d|n} \frac{\psi(d)}{d^{s_1+s_2-w}} \sum_{e|d} \frac{\mu(e)\chi(e)}{e^w} \right. \\ \left. \sum_{\substack{f=1 \\ (f, n/d)=1}} \frac{\mu(f)\chi(f)\psi(f)}{f^{s_1+2s_2-w}} \sum_{\substack{c_1=1 \\ (c_1, n/d)=1}} \frac{\psi(c_1)}{c_1^{s_1+s_2-w}} \right] dw ds_1 ds_2.$$

Invert the d - and n -sum to get

$$(6.21) \quad \frac{\sqrt{pq}\psi(p)\chi(-q)}{(pq)^2} \left(\frac{1}{2\pi i}\right)^3 \int_{(s_1)} \int_{(s_2)} \int_{(w)} \frac{\Gamma(w)\Gamma(s_2-w)}{\Gamma(s_2)} \frac{(pq)^{s_1+s_2}}{p^{s_1+s_2-w} q^{s_2}} \\ \left[L(\chi, w) \sum_{d=1} \frac{1}{d^{s_1+s_2}} \sum_{e|d} \frac{\mu(e)\chi(e)}{e^w} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0 \\ dn \ll (pq)^\theta L^2}} \frac{\widetilde{H}_{dn}(s_1, s_2)}{n^w} \sum_{r(dn)} \frac{1}{r^{s_2-w}} e\left(\frac{d^\circ mr + \frac{l_1 l_2 m}{d^\circ k^2} \bar{r}}{dn}\right) \overline{\psi(n)} \right. \\ \left. \sum_{\substack{f=1 \\ (f, n)=1}} \frac{\mu(f)\chi(f)\psi(f)}{f^{s_1+2s_2-w}} \sum_{\substack{c_1=1 \\ (c_1, n)=1}} \frac{\psi(c_1)}{c_1^{s_1+s_2-w}} \right] dw ds_1 ds_2.$$

Similarly, invert the e - and d -sums to get

$$(6.22) \quad \frac{\sqrt{pq}\psi(p)\chi(-q)}{(pq)^2} \left(\frac{1}{2\pi i}\right)^3 \int_{(s_1)} \int_{(s_2)} \int_{(w)} \frac{\Gamma(w)\Gamma(s_2-w)}{\Gamma(s_2)} \frac{(pq)^{s_1+s_2}}{p^{s_1+s_2-w}q^{s_2}} \\ \left[L(\chi, w) \sum_{e=1} \frac{\mu(e)\chi(e)}{e^{w+s_1+s_2}} \sum_{d=1} \frac{1}{d^{s_1+s_2}} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0 \\ edn \ll (pq)^\theta L^2}} \frac{\widetilde{H_{edn}}(s_1, s_2)}{n^w} \sum_{r(edn)} \frac{1}{r^{s_2-w}} e\left(\frac{d^\circ mr + \frac{l_1 l_2 m}{d^\circ k^2} \bar{r}}{edn}\right) \overline{\psi(n)} \right. \\ \left. \sum_{\substack{f=1 \\ (f,n)=1}} \frac{\mu(f)\chi(f)\psi(f)}{f^{s_1+2s_2-w}} \sum_{\substack{c_1=1 \\ (c_1,n)=1}} \frac{\psi(c_1)}{c_1^{s_1+s_2-w}} \right] dw ds_1 ds_2.$$

Finally, we remove the relatively prime conditions on the f - and c_1 -sums

$$(6.23) \quad \frac{\sqrt{pq}\psi(p)\chi(-q)}{(pq)^2} \left(\frac{1}{2\pi i}\right)^3 \int_{(s_1)} \int_{(s_2)} \int_{(w)} \frac{\Gamma(w)\Gamma(s_2-w)}{\Gamma(s_2)} \frac{(pq)^{s_1+s_2}}{p^{s_1+s_2-w}q^{s_2}} \\ \left[L(\chi, w) \sum_{e=1} \frac{\mu(e)\chi(e)}{e^{w+s_1+s_2}} \sum_{d=1} \frac{1}{d^{s_1+s_2}} \sum_{a=1} \frac{\mu(a)}{a^{s_1+s_2}} \sum_{b=1} \frac{\mu^2(b)\chi(b)}{b^{s_1+2s_2}} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0 \\ baedn \ll (pq)^\theta L^2}} \frac{\widetilde{H_{baedn}}(s_1, s_2)}{n^w} \right. \\ \left. \sum_{r(baedn)} \frac{1}{r^{s_2-w}} e\left(\frac{d^\circ mr + \frac{l_1 l_2 m}{d^\circ k^2} \bar{r}}{baedn}\right) \overline{\psi(n)} \sum_{f=1} \frac{\mu(f)\chi(f)\psi(f)}{f^{s_1+2s_2-w}} \sum_{c_1=1} \frac{\psi(c_1)}{c_1^{s_1+s_2-w}} \right] dw ds_1 ds_2.$$

This can be simplified to

$$(6.24) \quad \frac{\sqrt{pq}\psi(p)\chi(-q)}{(pq)^2} \left(\frac{1}{2\pi i}\right)^3 \int_{(s_1)} \int_{(s_2)} \int_{(w)} \frac{\Gamma(w)\Gamma(s_2-w)}{\Gamma(s_2)} \frac{(pq)^{s_1+s_2}}{p^{s_1+s_2-w}q^{s_2}} \\ \frac{L(\chi, w)L(\psi, s_1+s_2-w)}{L(\chi\psi, s_1+2s_2-w)} \left[\sum_{e=1} \frac{\mu(e)\chi(e)}{e^{w+s_1+s_2}} \sum_{d=1} \frac{1}{d^{s_1+s_2}} \sum_{a=1} \frac{\mu(a)}{a^{s_1+s_2}} \sum_{b=1} \frac{\mu^2(b)\chi(b)}{b^{s_1+2s_2}} \right. \\ \left. \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0 \\ baedn \ll (pq)^\theta L^2}} \frac{\widetilde{H_{baedn}}(s_1, s_2)}{n^w} \sum_{r(baedn)} \frac{1}{r^{s_2-w}} e\left(\frac{d^\circ mr + \frac{l_1 l_2 m}{d^\circ k^2} \bar{r}}{baedn}\right) \overline{\psi(n)} \right] dw ds_1 ds_2.$$

□

Remember $\Re(s_1) = 2$, $\Re(s_2) = 1/2 + \delta$, and $\Re(w) = 1/2$ with $\delta > 0$ small. We pick up a main term by shifting s_2 to the line $\Re(s_2) = 1/2 - \delta$ with a residue term

$$(6.25) \quad \frac{\sqrt{pq}\psi(p)\chi(-q)}{(pq)^2} \left(\frac{1}{2\pi i}\right)^2 \int_{(s_1)=2} \int_{(s_2)=1/2} \frac{(pq)^{s_1+s_2}}{p^{s_1}q^{s_2}} \\ \frac{L(\chi, s_2)L(\psi, s_1)}{L(\chi\psi, s_1+s_2)} \left[\sum_{e=1} \frac{\mu(e)\chi(e)}{e^{s_1+2s_2}} \sum_{d=1} \frac{1}{d^{s_1+s_2}} \sum_{a=1} \frac{\mu(a)}{a^{s_1+s_2}} \sum_{b=1} \frac{\mu^2(b)\chi(b)}{b^{s_1+2s_2}} \right. \\ \left. \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0 \\ baedn \ll (pq)^\theta L^2}} \frac{\widetilde{H_{baedn}}(s_1, s_2)}{n^{s_2}} \sum_{r(baedn)} e\left(\frac{d^\circ mr + \frac{l_1 l_2 m}{d^\circ k^2} \bar{r}}{baedn}\right) \overline{\psi(n)} \right] ds_1 ds_2,$$

plus a remainder term

$$(6.26) \quad \frac{\sqrt{pq}\psi(p)\chi(-q)}{(pq)^2} \left(\frac{1}{2\pi i}\right)^3 \int_{(s_1)=2} \int_{(s_2)=1/2-\delta} \int_{(w)=1/2} \frac{\Gamma(w)\Gamma(s_2-w)}{\Gamma(s_2)} \frac{(pq)^{s_1+s_2}}{p^{s_1+s_2-w}q^{s_2}} \\ \frac{L(\chi, w)L(\psi, s_1+s_2-w)}{L(\chi\psi, s_1+2s_2-w)} \left[\sum_{e=1} \frac{\mu(e)\chi(e)}{e^{w+s_1+s_2}} \sum_{d=1} \frac{1}{d^{s_1+s_2}} \sum_{a=1} \frac{\mu(a)}{a^{s_1+s_2}} \sum_{b=1} \frac{\mu^2(b)\chi(b)}{b^{s_1+2s_2}} \right. \\ \left. \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0 \\ baedn \ll (pq)^\theta L^2}} \frac{\widetilde{H_{baedn}}(s_1, s_2)}{n^w} \sum_{r(baedn)} \frac{1}{r^{s_2-w}} e\left(\frac{d^\circ mr + \frac{l_1 l_2 m}{d^\circ k^2} \bar{r}}{baedn}\right) \overline{\psi(n)} \right] dw ds_1 ds_2.$$

Remark. The “remainder” term is actually bigger than the “main” residual term, but only by a fixed but arbitrarily small amount. We expect that this “remainder” term via some kind of symmetry (perhaps a functional equation) is smaller than the “main” term. We hope that something similar to the analysis in getting the main term of [So] will work in this case. Regardless, we focus on the “main” term first.

6.3. Including the m -sum. Now we want to include the m -sum from (6.10) into (6.25) and deal with the remainder later in Section 7. The term to consider is

$$(6.27) \quad \frac{\sqrt{pq}\psi(p)\chi(-q)}{(pq)^2} \sum_{m \geq 1} \frac{1}{m^{1/2}} F'_{M'}\left(\frac{d^\circ m}{pq}\right) \left(\frac{1}{2\pi i}\right)^3 \int_{(s_1)=2} \int_{(s_2)=1/2} \frac{(pq)^{s_1+s_2}}{p^{s_1}q^{s_2}} \\ \frac{L(\chi, s_2)L(\psi, s_1)}{L(\chi\psi, s_1+s_2)} \left[\sum_{e=1} \frac{\mu(e)\chi(e)}{e^{s_1+2s_2}} \sum_{d=1} \frac{1}{d^{s_1+s_2}} \sum_{a=1} \frac{\mu(a)}{a^{s_1+s_2}} \sum_{b=1} \frac{\mu^2(b)\chi(b)}{b^{s_1+2s_2}} \right. \\ \left. \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0 \\ baedn \ll (pq)^\theta L^2}} \frac{\widetilde{H_{baedn}}(s_1, s_2)}{n^{s_2}} \sum_{r(baedn)} e\left(\frac{d^\circ mr + \frac{l_1 l_2 m}{d^\circ k^2} \bar{r}}{baedn}\right) \overline{\psi(n)} \right] ds_1 ds_2.$$

Recall $\widetilde{H_{baedn}}(s_1, s_2)$ depends on m . Specifically,

$$(6.28) \quad \widetilde{H}_\mu(s_1, s_2) = \int_0^\infty \int_0^\infty e\left(\frac{xd^\circ m}{\mu y} + \frac{y \frac{l_1 l_2 m}{d^\circ k^2}}{\mu x}\right) \int_{-\infty}^\infty e\left(\frac{t\mu}{xy}\right) F_M(t) V\left(\frac{4\pi\sqrt{td^\circ m}}{\sqrt{pq}x}\right) W\left(\frac{4\pi\sqrt{t \frac{l_1 l_2 m}{d^\circ k^2}}}{\sqrt{pq}y}\right) x^{s_1-1} y^{s_2-1} \frac{dt}{t^{1/2}} dx dy.$$

Make a change of variables $x \rightarrow \frac{\sqrt{md^\circ}x}{\sqrt{pq}}$ and $y \rightarrow \frac{\sqrt{m\frac{l_1l_2}{d^\circ k^2}}y}{\sqrt{pq}}$ to get $\widetilde{H}_\mu(s_1, s_2)$ equal to

$$(6.29) \quad \left(\frac{l_1l_2m}{k^2}\right)^{\frac{s_1+s_2}{2}} \frac{1}{(pq)^{\frac{s_1+s_2}{2}}} \int_0^\infty \int_0^\infty e\left(\frac{xd^\circ k m}{\sqrt{l_1l_2}hy} + \frac{y(l_1l_2)^{3/2}m}{d^\circ k^3hx}\right) \\ \int_{-\infty}^\infty e\left(\frac{t\mu k}{\sqrt{l_1l_2}mxy}\right) F_M(t) V\left(\frac{4\pi\sqrt{t}}{x}\right) W\left(\frac{4\pi\sqrt{t}}{y}\right) x^{s_1-1} y^{s_2-1} \frac{dt}{t^{1/2}} dx dy.$$

To simplify notation and to focus on the m -variable define

$$(6.30) \quad A_{s_1, s_2}(z) := z^{\frac{s_1+s_2}{2}} \int_0^\infty \int_0^\infty e\left(\frac{xd^\circ k z}{\sqrt{l_1l_2}baedny} + \frac{y(l_1l_2)^{3/2}z}{d^\circ k^3baednx}\right) \times \\ \int_{-\infty}^\infty e\left(\frac{tbaednk}{\sqrt{l_1l_2}zxy}\right) F_M(t) V\left(\frac{4\pi\sqrt{t}}{x}\right) W\left(\frac{4\pi\sqrt{t}}{y}\right) x^{s_1-1} y^{s_2-1} \frac{dt}{t^{1/2}} dx dy.$$

We interchange the integrals and the e, d, a, b -sums with the m -sum. This is ok as the m -sum is finite. So for the bracketed term of (6.27) along with $A_{s_1, s_2}(m)$ (which is just $\widetilde{H}_n(s_1, s_2)$ relabelled) and the m -sum we have

$$(6.31) \quad \sum_{e=1} \frac{\mu(e)\chi(e)}{e^{s_1+2s_2}} \sum_{d=1} \frac{1}{d^{s_1+s_2}} \sum_{a=1} \frac{\mu(a)}{a^{s_1+s_2}} \sum_{b=1} \frac{\mu^2(b)\chi(b)}{b^{s_1+2s_2}} \\ \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0 \\ baedn \ll (pq)^\theta L^2}} \frac{1}{n^{s_2}} \sum_{m \geq 1} \frac{1}{m^{1/2}} F'_{M'}\left(\frac{d^\circ m}{pq}\right) \sum_{r(baedn)} e\left(\frac{d^\circ mr + \frac{l_1l_2m}{d^\circ k^2}\overline{r}}{baedn}\right) A_{s_1, s_2}(m).$$

Performing Poisson summation in m modulo $baedn$ gives

$$(6.32) \quad \frac{1}{baedn} \sum_m \nu_{d^\circ}(baedn, m, \frac{l_1l_2}{d^\circ k^2}) \int_{-\infty}^\infty F'_{M'}\left(\frac{d^\circ v}{pq}\right) A_{s_1, s_2}(v) e\left(\frac{-mv}{baedn}\right) \frac{dv}{v^{1/2}},$$

where $\nu_b(n, m, a) := \#\{x : ax^2 - mx + b \equiv 0(n)\}$.

Following [R], $\nu_b(n, m, a)$ is multiplicative in n , and we can reduce the study of it to prime power modulus p^k . Suppose first we take $n = p$, p odd and $(a, p) = 1$, then completing the square

$$\nu_b(p, m, a) = \#\{x : x^2 \equiv (m\overline{2a})^2 - b\overline{a}(p)\}.$$

Again as in [R] using Hensel's lemma gives $\nu_b(p^k, m, a) = \nu_b(p, m, a)$ for $k > 1$. Then its clear

$$(6.33) \quad \nu_b(p, m, a) = 1 + \left(\frac{m^2 - 4ba}{p}\right) = \begin{cases} 2 & \text{if } m^2 - 4ba \equiv z^2(p) \\ 0 & \text{otherwise.} \end{cases}$$

Using multiplicativity, we have

$$(6.34) \quad \nu_{d^\circ}(\alpha\beta, m, \frac{l_1l_2}{d^\circ k^2}) = \nu_{d^\circ}(\mu^2(\alpha\beta), m, \frac{l_1l_2}{d^\circ k^2}) = \nu_{d^\circ}(\mu^2(\alpha), m, \frac{l_1l_2}{d^\circ k^2}) \nu_{d^\circ}(\mu^2(\beta), m, \frac{l_1l_2}{d^\circ k^2}),$$

where μ is the Mobius function. The last equality implies we can extend this result to square free modulus for $\nu_{d^\circ}(baedn, m, \frac{l_1l_2}{d^\circ k^2})$.

6.3.1. *Isolating the b, a, e, d, n -sums.* In order to isolate these sums, we open up $A_{s_1, s_2}(v)$ and focus on the terms containing b, a, e, d , and n . Let

$$(6.35) \quad T_{m,v}(c) := e\left(\frac{-mv}{c}\right) \int_0^\infty \int_0^\infty e\left(\frac{xd^{\circ 2}kv}{\sqrt{l_1 l_2} cy} + \frac{y(l_1 l_2)^{3/2}v}{d^{\circ 2} k^3 cx}\right) \\ \int_{-\infty}^\infty e\left(\frac{tck}{\sqrt{l_1 l_2} vxy}\right) F_M(t) V\left(\frac{4\pi\sqrt{t}}{x}\right) W\left(\frac{4\pi\sqrt{t}}{y}\right) x^{s_1-1} y^{s_2-1} \frac{dt}{t^{1/2}} dx dy.$$

Putting (6.35) back into (6.31), using the multiplicativity of (6.34) above, and interchanging the m -sum (after Poisson summation) with the other sums gives

$$(6.36) \quad \sum_m \int_{-\infty}^\infty F'_{M'}\left(\frac{d^{\circ}v}{pq}\right) \left[\sum_{e=1} \frac{\mu(e)\chi(e)\nu_{d^{\circ}}(e, m, \frac{l_1 l_2}{d^{\circ} k^2})}{e^{s_1+2s_2+1}} \sum_{d=1} \frac{\nu_{d^{\circ}}(d, m, \frac{l_1 l_2}{d^{\circ} k^2})}{d^{s_1+s_2+1}} \sum_{a=1} \frac{\mu(a)\nu_{d^{\circ}}(a, m, \frac{l_1 l_2}{d^{\circ} k^2})}{a^{s_1+s_2+1}} \right. \\ \left. \sum_{b=1} \frac{\mu^2(b)\chi(b)\nu_{d^{\circ}}(b, m, \frac{l_1 l_2}{d^{\circ} k^2})}{b^{s_1+2s_2+1}} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0 \\ baedn \ll (pq)^{\theta} L^2}} \frac{\overline{\psi(n)}\nu_{d^{\circ}}(n, m, \frac{l_1 l_2}{d^{\circ} k^2}) T_{m,v}(baedn)}{n^{s_2+1}} \right] v^{\frac{s_1+s_2-1}{2}} dv.$$

To completely isolate the b, a, e, d, n -sums in the archimedean functions requires applying Mellin inversion to $T_{m,v}(c)$. It is easy to check by integration by parts in the t -variable that

$$T_{m,v}(c) \ll \frac{1}{(1 + |\Im(c)|)^N}$$

for any $N > 0$. The variable of the Mellin inversion is w and we integrate on the line $\Re(w) = \epsilon$, for $\epsilon > 0$ small. Using the above estimate on $T_{m,v}(c)$ we move the e, d, a, b, n -sums inside of the w -integral to get

$$(6.37) \quad \sum_m \int_{-\infty}^\infty F'_{M'}\left(\frac{d^{\circ}v}{pq}\right) \left(\frac{1}{2\pi i} \int_{(w)=\epsilon} \widetilde{T_{m,v}(w)} \left[\sum_{e=1} \frac{\mu(e)\chi(e)\nu_{d^{\circ}}(e, m, \frac{l_1 l_2}{d^{\circ} k^2})}{e^{s_1+2s_2+1+w}} \sum_{d=1} \frac{\nu_{d^{\circ}}(d, m, \frac{l_1 l_2}{d^{\circ} k^2})}{d^{s_1+s_2+1+w}} \right. \right. \\ \left. \sum_{a=1} \frac{\mu(a)\nu_{d^{\circ}}(a, m, \frac{l_1 l_2}{d^{\circ} k^2})}{a^{s_1+s_2+1+w}} \sum_{b=1} \frac{\mu^2(b)\chi(b)\nu_{d^{\circ}}(b, m, \frac{l_1 l_2}{d^{\circ} k^2})}{b^{s_1+2s_2+1+w}} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0 \\ baedn \ll (pq)^{\theta} L^2}} \frac{\overline{\psi(n)}\nu_{d^{\circ}}(n, m, \frac{l_1 l_2}{d^{\circ} k^2})}{n^{s_2+1+w}} \right] dw \right) v^{\frac{s_1+s_2-1}{2}} dv.$$

Let $\Delta := m^2 - 4\frac{l_1 l_2}{k^2}$, and $S := \{\mathfrak{p} \text{ prime} : \mathfrak{p} \mid 2q\Delta\}$. As the usual notation for primes is “ p ”, we use *mathfrak{p}* for primes of our Euler product below. Hopefully this does not cause too much confusion. Either way, this possible confusion of notation ends at in this section. Following [R], for a fixed prime $\mathfrak{p} \notin S$, we have

$$(6.38) \quad 1 + \sum_{k=1}^\infty \frac{\nu_{d^{\circ}}(\mathfrak{p}, m, \frac{l_1 l_2}{d^{\circ} k^2}) \overline{\psi}(\mathfrak{p}^k)}{\mathfrak{p}^{kr}} = 1 + \frac{(1 + (\frac{\Delta}{\mathfrak{p}})) \frac{\overline{\psi}(\mathfrak{p})}{\mathfrak{p}^r}}{(1 - \frac{\overline{\psi}(\mathfrak{p})}{\mathfrak{p}^r})} = \frac{(1 + \frac{\overline{\psi}(\mathfrak{p})(\frac{\Delta}{\mathfrak{p}})})}{(1 - \frac{\overline{\psi}(\mathfrak{p})}{\mathfrak{p}^r})}.$$

Now look at the individual sums in (6.37). Note with a negligible error we can remove the restriction

$$baedn \ll (pq)^{\theta} L^2.$$

Therefore, the e -sum following what we did for (6.38) can be written as an Euler product equaling

$$(6.39) \quad \prod_{\mathfrak{p} \notin S} \frac{(1 - \frac{\chi(\mathfrak{p})}{\mathfrak{p}^{s_1+2s_2+w}})}{(1 + \frac{\chi(\mathfrak{p})(\frac{\Delta}{\mathfrak{p}})}{\mathfrak{p}^{s_1+2s_2+w}})} \prod_{\mathfrak{p} \in S} \frac{1}{L_{\mathfrak{p},\Delta}(\chi, s_1 + 2s_2 + w)}.$$

Here $L_{\mathfrak{p},\Delta}(\chi, s)$ we do not make specific, but note it is bounded in the range $\Re(s) \geq 1$ by $(\Delta pq)^\epsilon$ using (6.54).

Notice the a -sum and d -sum cancel by standard Euler product arguments.

The n -sum equals

$$(6.40) \quad \prod_{\mathfrak{p} \notin S} \frac{(1 + \frac{\bar{\psi}(\mathfrak{p})(\frac{\Delta}{\mathfrak{p}})}{\mathfrak{p}^{s_2+1+w}})}{(1 - \frac{\bar{\psi}(\mathfrak{p})}{\mathfrak{p}^{s_2+1+w}})} \prod_{\mathfrak{p} \in S} L_{\mathfrak{p},\Delta}(\bar{\psi}, s_2 + 1 + w).$$

The b -sum is more difficult and its Euler product equals

$$(6.41) \quad \prod_{\mathfrak{p} \notin S} \frac{(1 - \frac{\chi^2(\mathfrak{p})(1+(\frac{\Delta}{\mathfrak{p}}))^2}{\mathfrak{p}^{2s_1+4s_2+1+2w}})}{(1 - \frac{\chi(\mathfrak{p})(1+(\frac{\Delta}{\mathfrak{p}})}{\mathfrak{p}^{s_1+2s_2+1+w}})} \prod_{\mathfrak{p} \in S} B_{\mathfrak{p},\Delta}(s_1 + 2s_2 + 1 + w) =: B(s_1 + 2s_2 + 1 + w).$$

Here we do not explicate $B_{\mathfrak{p},\Delta}(s)$ but analogous analysis to that of the e -sum we have

$$\prod_{\mathfrak{p} \in S} B_{\mathfrak{p},\Delta}(s) \ll (\Delta pq)^\epsilon$$

for $\Re(s) \geq 1$.

The analytic behavior of the e -sum (6.39) is the same as

$$\frac{L(s_1 + 2s_2 + 1 + w, (\frac{\Delta}{\cdot})\chi)}{L(s_1 + 2s_2 + 1 + w, \chi)}$$

up to the factors $p \in S$. Likewise, the n -sum (6.40) is the same as

$$\frac{L(s_2 + 1 + w, \bar{\psi})}{L(s_2 + 1 + w, (\frac{\Delta}{\cdot})\bar{\psi})}$$

and the b -sum is absolutely convergent for $\Re(s_1 + 2s_2 + w) > 1$. We will not address further the analytic structure of the b -sum. Label the b -sum $B(s)$.

We consolidate what we have from Proposition 6.8 and the Poisson summation on the m -sum getting

$$(6.42) \quad \frac{\sqrt{pq}\psi(p)\chi(-q)}{(pq)^2} \sum_m (\frac{1}{2\pi i})^2 \int_{(s_1)=2} \int_{(s_2)=1/2} (\frac{l_1 l_2}{k^2})^{\frac{s_1+s_2}{2}} \int_{(w)=\epsilon} \int_{-\infty}^{\infty} F'_{M'}(\frac{d^\circ v}{pq}) \widetilde{T_{m,v}}(w) \frac{1}{(pq)^{\frac{s_1+s_2}{2}}} \frac{(pq)^{s_1+s_2}}{p^{s_1} q^{s_2}} \\ \frac{L(s_2, \chi)L(s_1, \psi)}{L(s_1 + s_2, \chi\psi)} \frac{L(s_1 + 2s_2 + 1 + w, (\frac{\Delta}{\cdot})\chi)}{L(s_1 + 2s_2 + 1 + w, \chi)} \frac{L(s_2 + 1 + w, \bar{\psi})}{L(s_2 + 1 + w, (\frac{\Delta}{\cdot})\bar{\psi})} B(s_1 + 2s_2 + 1 + w) v^{\frac{s_1+s_2-1}{2}} dv dw ds_1 ds_2 \\ + O((pq)^{-A}).$$

The finite product of $L_{\mathfrak{p},\Delta}(s)$ for $\mathfrak{p} \in S$ in the above cases do not affect convergence issues. In other words, we can shift the contour of s_1, s_2 in (6.27) by looking at these quotients of L-functions above.

We aim to bound (6.42) now.

6.4. Bounding the v -integral. The v -integral is

$$(6.43) \quad \int_{-\infty}^{\infty} F'_{M'}\left(\frac{d^\circ v}{pq}\right) \widetilde{T_{m,v}}(w) v^{\frac{s_1+s_2-1}{2}} dv = \int_0^\infty \int_{-\infty}^\infty F'_{M'}\left(\frac{d^\circ v}{pq}\right) e\left(\frac{-mv}{h}\right) \int_0^\infty \int_0^\infty e\left(\frac{xd^\circ k v}{\sqrt{l_1 l_2} h y} + \frac{y(l_1 l_2)^{3/2} v}{d^\circ k^3 h x}\right) \\ \int_{-\infty}^\infty e\left(\frac{thk}{\sqrt{l_1 l_2} v x y}\right) F_M(t) V\left(\frac{4\pi\sqrt{t}}{x}\right) W\left(\frac{4\pi\sqrt{t}}{y}\right) \frac{dt}{t^{1/2}} x^{s_1-1} y^{s_2-1} dx dy h^{w-1} dh v^{\frac{s_1+s_2-1}{2}} dv.$$

Change variables $h \rightarrow hv$, $x \rightarrow \sqrt{t}x$ and $y \rightarrow \sqrt{t}y$ we can write (6.43) as

$$(6.44) \quad \int_{-\infty}^\infty F'_{M'}\left(\frac{d^\circ v}{pq}\right) \widetilde{T_{m,v}}(w) v^{\frac{s_1+s_2}{2}} dv = \left[\int_0^\infty F_M(t) t^{\frac{s_1+s_2-1}{2}} dt \right] \left[\int_{-\infty}^\infty F'_{M'}\left(\frac{d^\circ v}{pq}\right) v^{\frac{s_1+s_2+2w-1}{2}} dv \right] \times \\ \int_0^\infty \int_0^\infty V\left(\frac{4\pi}{x}\right) W\left(\frac{4\pi}{y}\right) \left(\int_0^\infty e\left(\frac{-m}{h}\right) e\left(\frac{xd^\circ k}{\sqrt{l_1 l_2} h y} + \frac{y(l_1 l_2)^{3/2}}{d^\circ k^3 h x}\right) e\left(\frac{hk}{\sqrt{l_1 l_2} x y}\right) h^{w-1} dh \right) x^{s_1-1} y^{s_2-1} dx dy.$$

Note that the v -integral is now bounded by $(pq)^\epsilon$ for any $\epsilon > 0$ with $s_1 = s_2 = 1/2$. The x, y , and t integrals are bounded and so we focus on the h -integral.

We completely isolate the test functions $F_M, F'_{M'}$ by changing $v \rightarrow \frac{pqv}{d^\circ}$ getting (6.44) equaling

$$(6.45) \quad \left(\frac{pq}{d^\circ}\right)^{\frac{s_1+s_2+2w+1}{2}} \left[\int_0^\infty F_M(t) t^{\frac{s_1+s_2-1}{2}} dt \right] \left[\int_{-\infty}^\infty F'_{M'}(v) v^{\frac{s_1+s_2+2w-1}{2}} dv \right] \int_0^\infty \int_0^\infty V\left(\frac{4\pi}{x}\right) W\left(\frac{4\pi}{y}\right) \\ \left(\int_0^\infty e\left(\frac{-m}{h}\right) e\left(\frac{xd^\circ k}{\sqrt{l_1 l_2} h y} + \frac{y(l_1 l_2)^{3/2}}{d^\circ k^3 h x}\right) e\left(\frac{hk}{\sqrt{l_1 l_2} x y}\right) h^{w-1} dh \right) x^{s_1-1} y^{s_2-1} dx dy.$$

Remark. So note the choice of $F_M, F'_{M'}$ from the approximate functional equation in (5.2) is independent of the analysis needed in studying the subconvexity problem for the Rankin-Selberg L-function. Therefore if we want, we can bring the M -sum back in and remove the partition of unity.

We will eventually need an estimate for the m -sum to converge. Changing variables from $h \rightarrow h^{-1}$ and integrating by parts j -times easily gives the bound for the h -integral

$$(6.46) \quad \int_0^\infty e\left(\frac{-m}{h}\right) e\left(\frac{xd^\circ k}{\sqrt{l_1 l_2} h y} + \frac{y(l_1 l_2)^{3/2}}{d^\circ k^3 h x}\right) e\left(\frac{hk}{\sqrt{l_1 l_2} x y}\right) h^{w-1} dh \ll \left(\frac{(l_1 l_2)^{3/2}}{d^\circ k m}\right)^j.$$

Define

$$A := A(x, y, d^\circ, k, l_1, l_2) := \left(\frac{xd^\circ k}{\sqrt{l_1 l_2} y} + \frac{y(l_1 l_2)^{3/2}}{d^\circ k^3 x} - m \right)$$

and

$$B := B(x, y, k, l_1, l_2) := \frac{k}{\sqrt{l_1 l_2} x y}.$$

We need to bound the integral,

$$(6.47) \quad \int_0^\infty e\left(\frac{A}{h} + Bh\right) h^{w-1} dh$$

in terms of pq . We write the h -integral in terms of Bessel functions by recalling the following identities from [IK],

$$(6.48) \quad \int_0^\infty \cos\left(\frac{x}{2}\left(y + \frac{1}{y}\right)\right) y^{s-1} dy = -\pi J_s(x) \sin(\pi s/2) - \pi Y_s(x) \cos(\pi s/2) = -\pi \frac{J_s(x) - J_{-s}(x)}{2 \sin(\pi s/2)},$$

and

$$(6.49) \quad \int_0^\infty \sin\left(\frac{x}{2}\left(y + \frac{1}{y}\right)\right) y^{s-1} dy = -\pi J_s(x) \cos(\pi s/2) - \pi Y_s(x) \sin(\pi s/2) = -\pi \frac{J_s(x) + J_{-s}(x)}{2 \cos(\pi s/2)}.$$

So

$$(6.50) \quad \int_0^\infty e\left(\frac{A}{h} + Bh\right) h^{w-1} dh = -\pi \left(\frac{A}{B}\right)^{w/2} \left[\frac{J_w(4\pi\sqrt{\frac{A}{B}}) - J_{-w}(4\pi\sqrt{\frac{A}{B}})}{2 \sin(\pi w/2)} + \frac{J_w(4\pi\sqrt{\frac{A}{B}}) + J_{-w}(4\pi\sqrt{\frac{A}{B}})}{2 \cos(\pi w/2)} \right].$$

Define

$$(6.51) \quad C(s_1, s_2, w) := \frac{L(s_2, \chi) L(s_1, \psi)}{L(s_1 + s_2, \chi\psi)} \frac{L(s_1 + 2s_2 + 1 + w, (\frac{\Delta}{\cdot})\chi)}{L(s_1 + 2s_2 + 1 + w, \chi)} \frac{L(s_2 + 1 + w, \bar{\psi})}{L(s_2 + 1 + w, (\frac{\Delta}{\cdot})\bar{\psi})} B(s_1 + 2s_2 + 1 + w)$$

as well as

$$(6.52) \quad D_{\frac{A}{B}}(s_1, s_2, w) = -\pi \int_0^\infty \int_0^\infty V\left(\frac{4\pi}{x}\right) W\left(\frac{4\pi}{y}\right) \left(\frac{A}{B}\right)^{w/2} \left[\frac{J_w(4\pi\sqrt{\frac{A}{B}}) - J_{-w}(4\pi\sqrt{\frac{A}{B}})}{2 \sin(\pi w/2)} + \frac{J_w(4\pi\sqrt{\frac{A}{B}}) + J_{-w}(4\pi\sqrt{\frac{A}{B}})}{2 \cos(\pi w/2)} \right] x^{s_1-1} y^{s_2-1} dx dy.$$

Then incorporating (6.51) and (6.52) into (6.42) gives

$$(6.53) \quad \frac{\psi(p)\chi(-q)}{pq} \sum_m \left(\frac{1}{2\pi i}\right)^2 \int_{(s_1)=2} \int_{(s_2)=1/2} \frac{(pq)^{s_1+s_2}}{p^{s_1} q^{s_2}} \left(\frac{l_1 l_2}{k^2}\right)^{\frac{s_1+s_2}{2}} \times \\ \int_{(w)=\epsilon} \left(\frac{pq}{d^\circ}\right)^w \left[\int_0^\infty F_M(t) t^{\frac{s_1+s_2-1}{2}} dt \right] \left[\int_{-\infty}^\infty F'_{M'}(v) v^{\frac{s_1+s_2+2w-1}{2}} dv \right] \times \\ D_{\frac{A}{B}}(s_1, s_2, w) C(s_1, s_2, w) dw ds_1 ds_2 + O((pq)^{-A}).$$

We now make a contour shift on the line $\Re(s_1) = 2$ to $\Re(s_1) = 1/2$. We encounter no poles in this shift.

Now write $s_1 = 1/2 + it_1$, $s_2 = 1/2 + it_2$, and $w = iy$. We now bound the w -integral. First

$$C(1/2 + it_1, 1/2 + it_2, iy) \ll (|t_1|p)^{\theta_1} (|t_2|q)^{\theta_2} (|t_1 t_2 y|pq)^\epsilon,$$

where θ_1, θ_2 are parameters $\leq 1/4$. Here we are using

$$(6.54) \quad \left(\frac{1}{|t|q}\right)^\epsilon \ll L(\sigma + it, \chi) \ll (|t|q)^\epsilon,$$

for χ a character of modulus q , $\sigma \geq 1$ and $t \in \mathbb{R}$. We also are using the (sub)convexity bound

$$L(1/2 + it, \chi) \ll (|t|q)^{\theta+\epsilon}$$

with $\theta \leq 1/4$ [IK].

Second, we bound

$$(6.55) \quad D_{\frac{A}{B}}(1/2 + it_1, 1/2 + it_2, iy).$$

Proposition 6.9.

$$(6.56) \quad D_{\frac{A}{B}}(1/2 + it_1, 1/2 + it_2, i\gamma) \ll (t_1 t_2 \gamma \frac{A}{B})^\epsilon$$

Proof. As the x, y integrals are bounded, we first look at the case $w = i\gamma \neq 0$, it is sufficient to look at

$$(6.57) \quad \left(\frac{A}{B}\right)^{i\gamma/2} \frac{J_{2i\gamma}\left(\frac{4\pi\sqrt{A}}{\sqrt{B}}\right)}{\sin(\pi i\gamma/2)} \ll \frac{J_{2i\gamma}\left(\frac{4\pi\sqrt{A}}{\sqrt{B}}\right)}{\sinh(\pi\gamma/2)}.$$

Let $T(x, y) = \frac{4\pi\sqrt{A}}{\sqrt{B}}$. Fix $w = i\gamma \neq 0$, then we break the analysis into two cases: $|\gamma| \geq T(x, y)$ and $|\gamma| \leq T(x, y)$.

First we study the Bessel asymptotic for $T(x, y)$ smaller than the index. We need the behavior for $J_{2i\gamma}(T)$ for $\gamma \geq cT(x, y)$ and c any positive number. Let $z = \sqrt{T^2 + \gamma^2}$, then from ([E], pg.87)

$$(6.58) \quad J_{2i\gamma}(T) = (2\pi^{1/2})^{-1} z^{-1/2} e^{\frac{-\pi i}{4}} \exp(\pi\gamma) e\left(\frac{z}{\pi} - \frac{\gamma}{\pi} \log\left(\frac{z - \gamma}{T}\right)\right) \times \\ \left\{ 1 + \frac{1}{2i\gamma} \left(\frac{\gamma}{8z} - \frac{5\gamma^3}{24z^3}\right) + \frac{1}{(2i\gamma)^2} \left(\frac{9\gamma^2}{128z^2} - \frac{231\gamma^4}{576z^4} + \dots\right) \right\}$$

Incorporating this asymptotic for the appropriate range of integration, the x, y -integral can be bounded up to a constant by

$$\int_0^\infty \int_0^\infty V\left(\frac{4\pi}{x}\right) W\left(\frac{4\pi}{y}\right) T(x, y)^w \frac{1}{w^{1/2} \sqrt{1 + (T(x, y)/w)^2}} x^{s_1-1} y^{s_2-1} dx dy + \text{remainder terms}.$$

The main term is trivially bounded by

$$\frac{\left|\frac{4\pi\sqrt{A}}{\sqrt{B}}\right|^{\Re(w)}}{|w|^{1/2}},$$

while the remainder terms are bounded by

$$\frac{\left|\frac{4\pi\sqrt{A}}{\sqrt{B}}\right|^{\Re(w)}}{|w|^{1/2+k}}$$

for $k \in \mathbb{N}$.

Now we study the asymptotic for the argument $T(x, y)$ bigger than the index. For $T \geq |\gamma|$, we use from ([Wat] pg. 205)

$$(6.59) \quad J_{2i\gamma}(T) = \frac{1}{\sqrt{2\pi T}} (W_1(2i\gamma, T) e^{iT} + W_2(2i\gamma, T) e^{-iT}),$$

where

$$\frac{\partial^{(j)}}{\partial x^j} W_i(2iy, x) \ll_j (1 + |x|)^{-j} \cosh(\pi y),$$

for $i = 1, 2, j \geq 0$.

So the x, y - integral, in this case, is bounded by

$$(6.60) \quad \int_0^\infty \int_0^\infty V\left(\frac{4\pi}{x}\right) W\left(\frac{4\pi}{y}\right) T(x, y)^{i\gamma-1/2} \left[\frac{W_1(2i\gamma, T(x, y))e^{iT(x, y)} + W_2(2i\gamma, T(x, y))e^{-iT(x, y)}}{\sinh(\pi\gamma)} \right] \times \\ x^{s_1-1} y^{s_2-1} dx dy \ll \frac{1}{\sqrt{T(x, y)}} \ll \frac{1}{|\gamma|^{1/4}}.$$

Without loss of generality this argument works for the other Bessel functions divided by the hyperbolic functions in (6.52).

Now for the case $w = 0$,

$$\lim_{w \rightarrow 0} \left[\frac{J_w(4\pi T) - J_{-w}(4\pi T)}{2 \sin(\pi w/2)} + \frac{J_w(4\pi T) + J_{-w}(4\pi T)}{2 \cos(\pi w/2)} \right] = \frac{Y_0(4\pi T)}{2} + J_0(4\pi T).$$

Here we use the continuity of the index of the Bessel function ([GR] 8.487) and the definition

$$Y_w(x) = \frac{J_w(x) \cos(\pi w) - J_{-w}(x)}{\sin(\pi w)}.$$

Assume $|T(x, y)|$ is larger than an absolute positive constant C . Then for the second term $J_0(4\pi T)$ we can use the same analysis as is used to get (6.60). For the first term we can use the asymptotic

$$Y_0(z) = \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4}\right) + O(|z|^{-1}),$$

[GR]. Similar to (6.60) we can bound the x, y -integral up to an absolute constant by $\frac{1}{\sqrt{T(x, y)}} \ll \frac{1}{\sqrt{C}}$.

If $|T(x, y)|$ is smaller than C , then we use the first term of the power series expansion for $J_0(4\pi T)$ and the asymptotic formula [GR]

$$Y_0(z) = \frac{2}{\pi} [\ln(z/2) + \gamma],$$

with γ Euler's constant. The integral in this case can be bounded by $\ln(2\pi T) \ll T^\epsilon$.

If $T(x, y) = 0$, the x, y integrals converge and are independent of the parameters p, q . □

6.5. Bounding the s_1, s_2, w -integrals. Recall we are trying to bound

$$(6.61) \quad \frac{\psi(p)\chi(-q)}{pq} \sum_m \left(\frac{1}{2\pi i}\right)^2 \int_{(s_1)=2} \int_{(s_2)=1/2} \frac{(pq)^{s_1+s_2}}{p^{s_1} q^{s_2}} \left(\frac{l_1 l_2}{k^2}\right)^{\frac{s_1+s_2}{2}} \times \\ \int_{(w)=\epsilon} \left(\frac{pq}{d^\circ}\right)^w \left[\int_0^\infty F_M(t) t^{\frac{s_1+s_2-1}{2}} dt \right] \left[\int_{-\infty}^\infty F_{M'}(v) v^{\frac{s_1+s_2+2w-1}{2}} dv \right] D_{\frac{A}{B}}(s_1, s_2, w) C(s_1, s_2, w) dw ds_1 ds_2 \\ + O((pq)^{-A}).$$

By the previous section we have a bound for the functions $D_{\frac{A}{B}}(s_1, s_2, w)$ and $C(s_1, s_2, w)$. We must now ensure the s_1, s_2 , and w -integrals converge.

Using integration by parts we have

$$\int_0^\infty F_{M'}(t) t^{\frac{s_1+s_2+2w-1}{2}} dt \ll \left(\frac{|s_1 + s_2|}{1 + |w|} \right)^M$$

for any $M > 0$. We also have

$$\int_0^\infty F_M(t) t^{\frac{s_1+s_2-1}{2}} dt \ll \frac{1}{(1 + |s_1 + s_2|)^N},$$

for any $N > 0$. Clearly, if we choose $N \geq M + 2 \geq 2$ all three of the integrals converge.

Remember the m -sum converges by the estimate (6.46). If we include the l_1, l_2 -sums into (6.53), we have

$$(6.62) \quad \frac{\psi(p)\chi(-q)}{pq} \sum_{l_1, l_2 \leq L} x_{l_1} x_{l_2} \sum_{k|(l_1, l_2)} \sum_{d^\circ | \frac{l_1 l_2}{k^2}} \frac{1}{d^{\circ 1/2}} \sum_m \left(\frac{1}{2\pi i} \right)^2 \int_{(s_1)=1/2} \int_{(s_2)=1/2} \frac{(pq)^{s_1+s_2}}{p^{s_1} q^{s_2}} \left(\frac{l_1 l_2}{k^2} \right)^{\frac{s_1+s_2}{2}} \times \\ \int_{(w)=\epsilon} \left(\frac{pq}{d^\circ} \right)^w \left[\int_0^\infty F_M(t) t^{\frac{s_1+s_2-1}{2}} dt \right] \left[\int_{-\infty}^\infty F'_{M'}(v) v^{\frac{s_1+s_2+2w-1}{2}} dv \right] \times \\ D(s_1, s_2, w) C(s_1, s_2, w) dw ds_1 ds_2 + O((pq)^{-A}) \\ \ll \frac{p^{\theta_1+\epsilon} q^{\theta_2+\epsilon}}{\sqrt{pq}} \sum_{l_1, l_2 \leq L} x_{l_1} x_{l_2} l_1^{1/2} l_2^{1/2} \sum_{k|(l_1, l_2)} \frac{1}{k} \sum_{d^\circ | \frac{l_1 l_2}{k^2}} \frac{1}{d^{\circ 1/2}} \ll \frac{p^{\theta_1+\epsilon} q^{\theta_2+\epsilon}}{\sqrt{pq}} L^{1+\epsilon} \|x\|_2^2$$

with $\theta_1, \theta_2 \leq 1/4$.

7. REMAINDER TERM FROM (6.26)

It turns out in this case the “remainder” term is bigger than the residue term with respect to pq (but only by a fixed but arbitrary amount), however the analysis is essentially the same. We hope that there is a hidden symmetry similar to [So] so that this “remainder” term cancels via some kind of functional equation and leaves the beautiful main term (6.53).

We study

$$(7.1) \quad \frac{\sqrt{pq}\psi(p)\chi(-q)}{(pq)^2} \left(\frac{1}{2\pi i} \right)^3 \int_{(s_1)=2} \int_{(s_2)=1/2-\delta} \int_{(\alpha)=1/2} \frac{\Gamma(\alpha)\Gamma(s_2-\alpha)}{\Gamma(s_2)} \frac{(pq)^{s_1+s_2}}{p^{s_1+s_2-\alpha} q^{s_2}} \\ \frac{L(\chi, \alpha) L(\psi, s_1 + s_2 - \alpha)}{L(\chi\psi, s_1 + 2s_2 - \alpha)} \left[\sum_{e=1} \frac{\mu(e)\chi(e)}{e^{\alpha+s_1+s_2}} \sum_{d=1} \frac{1}{d^{s_1+s_2}} \sum_{a=1} \frac{\mu(a)}{a^{s_1+s_2}} \sum_{b=1} \frac{\mu^2(b)\chi(b)}{b^{s_1+2s_2}} \right. \\ \left. \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0 \\ baedn \ll (pq)^\theta L^2}} \frac{\widetilde{\psi(n) H_{baedn}(s_1, s_2)}}{n^\alpha} \sum_{r(baedn)} \frac{1}{r^{s_2-\alpha}} e\left(\frac{d^\circ m r + \frac{l_1 l_2 m}{d^\circ k^2} \bar{r}}{baedn} \right) \right] dw ds_1 ds_2.$$

As in the main term we will execute Poisson summation in the m -sum modulo $baedn$; the analogous interchanges of sums and integrals is identical.

$$(7.2) \quad \frac{1}{baedn} \sum_m \sum_{\substack{r(baedn) \\ \{r: \frac{l_1 l_2 r^2}{d^\circ k^2} - mr + d^\circ \equiv 0 (baedn)\}}} \frac{1}{r^{s_2-\alpha}} \int_{-\infty}^\infty F'_{M'}\left(\frac{d^\circ v}{pq}\right) A_{s_1, s_2}(v) e\left(\frac{-mv}{baedn}\right) \frac{dv}{v^{1/2}}.$$

Applying the same steps up to (6.42) gives

$$(7.3) \quad \frac{\sqrt{pq}\psi(p)\chi(-q)}{(pq)^2} \sum_m \left(\frac{1}{2\pi i}\right)^3 \int_{(s_1)=2} \int_{(s_2)=1/2-\delta} \left(\frac{l_1 l_2}{k^2}\right)^{\frac{s_1+s_2}{2}} \int_{(\alpha)=1/2} \frac{\Gamma(\alpha)\Gamma(s_2-\alpha)}{\Gamma(s_2)} \frac{(pq)^{s_1}(pq)^{s_2}}{p^{s_1+s_2-\alpha}q^{s_2}} \\ \frac{L(\alpha, \chi)L(s_1+s_2-\alpha, \psi)}{L(s_1+2s_2-\alpha, \chi\psi)} \int_{(w)=\epsilon} \int_{-\infty}^{\infty} F'_{M'}\left(\frac{d^\circ v}{pq}\right) \widetilde{T_{m,v}}(w) \left[\sum_{e=1} \frac{\mu(e)\chi(e)}{e^{\alpha+s_1+2s_2+1+w}} \sum_{d=1} \frac{1}{d^{s_1+s_2+1+w}} \sum_{a=1} \frac{\mu(a)}{a^{s_1+s_2+1+w}} \right. \\ \left. \sum_{b=1} \frac{\mu^2(b)\chi(b)}{b^{s_1+2s_2+1+w}} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{\overline{\psi(n)}}{n^{\alpha+1+w}} \left(\sum_{\substack{r \in (baedn) \\ \{r: \frac{l_1 l_2 r^2}{d^\circ k^2} - mr + d^\circ \equiv 0 \pmod{baedn}\}}} \frac{1}{r^{s_2+w-\alpha}} \right) \right] v^{\frac{s_1+s_2-1}{2}} dv dw ds_1 ds_2 + O((pq)^{-A}).$$

We shift the $(s_1) = 2$ to the line $1/2 + \delta$. There are no poles encountered in this contour shift.

The same archimedean analysis as is done in Section 6.4 simplifies this term to

$$(7.4) \quad \frac{\sqrt{pq}\psi(p)\chi(-q)}{(pq)^2} \sum_m \left(\frac{1}{2\pi i}\right)^3 \int_{(s_1)=1/2+\delta} \int_{(s_2)=1/2-\delta} \left(\frac{l_1 l_2}{k^2}\right)^{\frac{s_1+s_2}{2}} \int_{(\alpha)=1/2} \frac{\Gamma(\alpha)\Gamma(s_2-\alpha)}{\Gamma(s_2)} \frac{(pq)^{s_1}(pq)^{s_2}}{p^{s_1+s_2-\alpha}q^{s_2}} \\ \frac{L(\alpha, \chi)L(s_1+s_2-\alpha, \psi)}{L(s_1+2s_2-\alpha, \chi\psi)} \int_{(w)=\epsilon} D(s_1, s_2, w) \left[\sum_{e=1} \frac{\mu(e)\chi(e)}{e^{\alpha+s_1+2s_2+1+w}} \sum_{d=1} \frac{1}{d^{s_1+s_2+1+w}} \sum_{a=1} \frac{\mu(a)}{a^{s_1+s_2+1+w}} \right. \\ \left. \sum_{b=1} \frac{\mu^2(b)\chi(b)}{b^{s_1+2s_2+1+w}} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{\overline{\psi(n)}}{n^{\alpha+1+w}} \left(\sum_{\substack{r \in (baedn) \\ \{r: \frac{l_1 l_2 r^2}{d^\circ k^2} - mr + d^\circ \equiv 0 \pmod{baedn}\}}} \frac{1}{r^{s_2+w-\alpha}} \right) \right] dw ds_1 ds_2 + O((pq)^{-A}),$$

with again $D(s_1, s_2, w)$ defined as in (6.52).

The r -sum has at most two terms in it following (6.33), so it is bounded by $2(baedn)^\delta$. The entire sum inside the brackets of (7.4) is absolutely convergent and is bounded (in terms of p, q) by $(pq)^\epsilon$. Using (6.54) gives

$$(7.5) \quad \frac{L(\alpha, \chi)L(s_1+s_2-\alpha, \psi)}{L(s_1+2s_2-\alpha, \chi\psi)} \ll p^{\theta_1+\epsilon} q^{\theta_2+\epsilon}$$

with $\theta_1, \theta_2 \leq 1/4$. The same analysis is done on $D(s_1, s_2, w)$ as in the previous section. The only difference between the analysis of this section and the previous section is the α -integral. We can use an application of Stirling's approximation to the ratio of Gamma functions as in [Y] to get

$$\frac{\Gamma(\alpha)\Gamma(s_2-\alpha)}{\Gamma(s_2)} \ll |1 + t_2(1 - \frac{t}{t_2})|^{-1/2}$$

where $\alpha = 1/2 + it$ and $s_2 = 1/2 - \delta + it_2$ with t, t_2 large. A standard integration by parts argument in the α -integral ensures that it converges.

So analogously to getting the bound (6.62), (7.4) is bounded by

$$(7.6) \quad \frac{p^{\theta_1} q^{\theta_2+\delta} (pq)^\epsilon}{\sqrt{pq}} \sum_{l_1, l_2 \leq L} x_{l_1} x_{l_2} l_1^{1-\delta/2} l_2^{1-\delta/2} \sum_{k|(l_1, l_2)} \frac{1}{k^{3/2-\delta/2}} \sum_{d^\circ | \frac{l_1 l_2}{k^2}} \frac{1}{d^{3/4}} \ll \frac{p^{\theta_1} q^{\theta_2+\delta} (pq)^\epsilon}{\sqrt{pq}} L^{1-2\delta+\epsilon} \|x\|_2^2$$

with $\theta_1, \theta_2 \leq 1/4$.

Combining (6.62) and (7.4) we get the Kloosterman-Kloosterman term with $n \neq 0$ is bounded by

$$(7.7) \quad \frac{p^{\theta_1+\epsilon} q^{\theta_2+\epsilon}}{\sqrt{pq}} L^{1+\epsilon} \|x\|_2^2 (1+q^\delta).$$

8. DELTA-DELTA TERM

The term from (5.6) that we now study is

$$(8.1) \quad \sum_{l_1, l_2 \leq L} x_{l_1} x_{l_2} \sum_{q|(l_1, l_2)} \sum_{d^\circ | \frac{l_1 l_2}{k^2}} \frac{1}{d^{\circ 1/2}} \sum_{m \geq 1} \frac{1}{m^{1/2}} F'_{M'}\left(\frac{d^\circ m}{pq}\right) \sum_{n \geq 1} \frac{1}{n^{1/2}} F_M\left(\frac{n}{pq}\right) D_{n, \frac{l_1 l_2 m}{d^\circ k^2}}(V) D_{n, d^\circ m}(W)$$

Now this term is non-zero only if $n = \frac{l_1 l_2 m}{d^\circ k^2}$ and $n = d^\circ m$, which implies $\sqrt{l_1 l_2} = q d^\circ$. So we can bound (8.1) by

$$(8.2) \quad \sum_{l_1, l_2 \leq L} x_{l_1} x_{l_2} \sum_{k|(l_1, l_2)} \sum_{\substack{d^\circ | \frac{l_1 l_2}{k^2} \\ \sqrt{l_1 l_2} = q d^\circ}} \frac{1}{d^\circ} \sum_{m \geq 1} \frac{1}{m} F'_{M'}\left(\frac{d^\circ m}{pq}\right) F_M\left(\frac{d^\circ m}{pq}\right).$$

Using a standard Mellin inversion we can reduce to

$$\left[\int_0^\infty F'_{M'}(t) F_M(t) \frac{dt}{t} \right] \sum_{l_1, l_2 \leq L} \frac{x_{l_1} x_{l_2}}{(l_1 l_2)^{1/2}} \sum_{k|(l_1, l_2)} k + O((pq)^{-M}),$$

for any $M > 0$.

An easy estimate for this last line is $(pq)^\epsilon L^\epsilon \|x\|_2^2$.

9. DELTA-KLOOSTERMAN TERM

It is obvious the analysis for this term will suffice for the Kloosterman-delta term. The term is

$$(9.1) \quad \sum_{l_1, l_2 \leq L} x_{l_1} x_{l_2} \sum_{k|(l_1, l_2)} \sum_{d^\circ | \frac{l_1 l_2}{k^2}} \frac{1}{d^{\circ 1/2}} \sum_{m \geq 1} \frac{1}{m^{1/2}} F'_{M'}\left(\frac{d^\circ m}{pq}\right) \sum_{n \geq 1} \frac{1}{n^{1/2}} F_M\left(\frac{n}{pq}\right) D_{n, \frac{l_1 l_2 m}{d^\circ k^2}}(V) \times \\ \left(\sum_{c_2 \equiv 0(q)} \frac{S_\psi(n, d^\circ m, c_2)}{c_2} W\left(\frac{4\pi \sqrt{nd^\circ m}}{c_2}\right) \right)$$

Similar to the previous section, the only non-zero term is when $n = \frac{l_1 l_2 m}{d^\circ k^2}$, which then reduces (9.1) to

$$(9.2) \quad \sum_{l_1, l_2 \leq L} \frac{x_{l_1} x_{l_2}}{(l_1 l_2)^{1/2}} \sum_{k|(l_1, l_2)} k \sum_{d^\circ | \frac{l_1 l_2}{k^2}} \sum_{m \geq 1} \frac{1}{m} F'_{M'}\left(\frac{d^\circ m}{pq}\right) F_M\left(\frac{\frac{l_1 l_2 m}{d^\circ k^2}}{pq}\right) \times \\ \left(\sum_{c_2 \equiv 0(q)} \frac{S_\psi(\frac{l_1 l_2 m}{d^\circ k^2}, d^\circ m, c_2)}{c_2} W\left(\frac{4\pi \sqrt{\frac{l_1 l_2 m^2}{k^2}}}{c_2}\right) \right)$$

This calculation is very similar to [R].

We isolate the m -sum to get

$$\sum_m \frac{1}{m} F'_{M'}\left(\frac{d^\circ m}{pq}\right) F_M\left(\frac{\frac{l_1 l_2 m}{d^\circ k^2}}{pq}\right) \sum_{y(c_2)^*} \psi(y) e\left(\frac{\frac{l_1 l_2 m y}{d^\circ k^2} + d^\circ m \bar{y}}{c_2}\right) W\left(\frac{4\pi \sqrt{\frac{l_1 l_2 m^2}{k^2}}}{c_2}\right).$$

This is very similar to (6.31) above. Performing Poisson summation on the m -sum modulo c_2 , and including the c_2 -sum we get

$$(9.3) \quad \sum_{c_2 \equiv 0(q)} \frac{1}{c_2} \sum_m \sum_{\substack{y(c_2)^* \\ \frac{l_1 l_2}{d^\circ k^2} y^2 - m y + d^\circ \equiv 0(c_2)}} \psi(y) \int_{-\infty}^{\infty} F'_{M'}\left(\frac{d^\circ t}{pq}\right) F_M\left(\frac{\frac{l_1 l_2 t}{d^\circ k^2}}{pq}\right) W\left(\frac{4\pi \frac{t}{k} \sqrt{l_1 l_2}}{c_2}\right) e\left(\frac{-mt}{c_2}\right) \frac{dt}{t}.$$

Let us change variables $t \rightarrow \frac{kt}{\sqrt{l_1 l_2}}$, giving

$$(9.4) \quad \sum_{c_2 \equiv 0(q)} \frac{1}{c_2} \sum_m \sum_{\substack{y(c_2)^* \\ \frac{l_1 l_2}{d^\circ k^2} y^2 - m y + d^\circ \equiv 0(c_2)}} \psi(y) \int_{-\infty}^{\infty} F'_{M'}\left(\frac{\frac{d^\circ kt}{\sqrt{l_1 l_2}}}{pq}\right) F_M\left(\frac{\frac{\sqrt{l_1 l_2} t}{d^\circ k}}{pq}\right) W\left(\frac{4\pi t}{c_2}\right) e\left(\frac{-mqt}{\sqrt{l_1 l_2} c_2}\right) \frac{dt}{t}.$$

We bound the y -sum by 2 as in Section 6.3 and define

$$T_m(x) := \int_{-\infty}^{\infty} F'_{M'}\left(\frac{d^\circ t}{pq}\right) F_M\left(\frac{\frac{l_1 l_2 t}{d^\circ k^2}}{pq}\right) W\left(\frac{4\pi \frac{t}{k} \sqrt{l_1 l_2}}{x}\right) e\left(\frac{-mt}{x}\right) \frac{dt}{t}.$$

Then interchanging the m -sum and c_2 -sum with another application of Mellin inversion, (9.3) is bounded by

$$(9.5) \quad 2 \sum_m \sum_{c_2 \equiv 0(q)} \frac{1}{c_2} T_m(c_2) = 2 \sum_m \left(\frac{1}{2\pi i}\right) \int_{(\sigma)} \widetilde{T}_m(s) \frac{\zeta(1+s)}{q^{1+s}} ds = \\ 2 \sum_m \left[\frac{\widetilde{T}_m(0)}{q} + \left(\frac{1}{2\pi i}\right) \int_{(-\delta)} \widetilde{T}_m(s) \frac{\zeta(1+s)}{q^{1+s}} ds \right],$$

where $\delta > 0$. The main term is

$$2 \sum_m \frac{1}{q} \int_0^\infty \int_{-\infty}^\infty F'_{M'}\left(\frac{d^\circ t}{pq}\right) F_M\left(\frac{\frac{l_1 l_2 t}{d^\circ k^2}}{pq}\right) W\left(\frac{4\pi \frac{t}{k} \sqrt{l_1 l_2}}{x}\right) e\left(\frac{-mt}{x}\right) \frac{dt}{t} \frac{dx}{x}$$

As $F_M, F'_{M'}, W$ are compactly supported $\sqrt{l_1 l_2} \sim q d^\circ$, the t -integral and x -integral is of bounded size (independent of p, q due to the multiplicative Haar measures). The m -sum converges by an integration by parts argument. Also the m -sum can be truncated at length $O(pq)$, with a remainder term having a bound $O((pq)^{-M})$, for any $M \in \mathbb{N}$. Integrate by parts once in the t -integral to gain a term of size $d^\circ \sim \frac{\sqrt{l_1 l_2}}{q}$. The size of the integral is of size $\frac{pq}{d^\circ}$, and so bounding the m trivially gets the whole main term is of size $O(p^\epsilon q^{\epsilon-1})$. The remainder term by an analogous argument gives $O(p^\epsilon q^{\epsilon-1})$ also.

Our leftover analysis is then

$$(9.6) \quad p^\epsilon q^{\epsilon-1} \sum_{l_1, l_2 \leq L} (l_1 l_2)^{-1/2} x_{l_1} x_{l_2} \sum_{q|(l_1, l_2)} q \ll p^\epsilon q^{\epsilon-1} L^\epsilon \|x\|_2^2.$$

Now one can apply the same argument as we did for the delta-Kloosterman term to the Kloosterman-delta term, and get analogous bound $O(q^\epsilon p^{\epsilon-1})$, and bounding the l_1, l_2 sum gives $q^\epsilon p^{\epsilon-1} L^\epsilon \|x\|_2^2$.

10. POSITIVITY OF TEST FUNCTIONS ON SPECTRAL SIDE OF TRACE FORMULA

In order to go from estimating

$$(10.1) \quad \sum_{f \in H(p, \chi)} \frac{1}{(f, f)} h(V, t_f) \left| \sum_{l \leq L} x_l \lambda_f(l) \right|^2 \sum_{h \in H(q, \psi)} \frac{1}{(h, h)} h(W, t_h) |L(1/2, f \times h)|^2$$

to a single term

$$(10.2) \quad \frac{1}{(f, f)(h, h)} h(V, t_f) h(W, t_h) \left| \sum_{l \leq L} x_l \lambda_f(l) \right|^2 |L(1/2, f \times h)|^2,$$

we need careful choices of our test functions V, W . We use the following proposition of [Venk1]:

Proposition 10.1. *Given $j_0 \in \mathbb{N}, \epsilon > 0$ and an integer $N > 0$, there is a V of compact support so that $h(V, t_{j_0}) = 1$, and for all $j' \neq j_0$, $h(V, t_{j'}) \ll \epsilon(1 + |t_{j'}|)^{-N}$, and for all k odd, $h(V, k) \ll \epsilon k^{-N}$.*

Given $k_0, \epsilon > 0$ and an integer $N > 0$, there is a V of compact support so that $h(V, k_0) = 1$, $h(V, k) \ll \epsilon k^{-N}$ for k odd $k \neq k_0$, and $h(V, t) \ll (1 + |t|)^{-N}$ for all \mathbb{R} .

Using the above proposition we can choose V, W to isolate spectral parameters $\{t_{f^\circ}, t_{h^\circ}\}$ with N_V, N_W large enough and $\epsilon = \epsilon_V = \epsilon_W$ small enough so that the sum

$$(10.3) \quad \sum_{\substack{f \in H(p, \chi) \\ t_f \neq t_{f^\circ}}} \frac{1}{(f, f)} h(V, t_f) \left| \sum_{l \leq L} x_l \lambda_f(l) \right|^2 \sum_{\substack{h \in H(q, \psi) \\ t_h \neq t_{h^\circ}}} \frac{1}{(h, h)} h(W, t_h) |L(1/2, f \times h)|^2 < \epsilon.$$

With this choice of V, W (10.2) we have the inequality

$$(10.4) \quad \frac{1}{(f^\circ, f^\circ)(h^\circ, h^\circ)} \left| \sum_{l \leq L} x_l \lambda_f(l) \right|^2 |L(1/2, f^\circ \times h^\circ)|^2 \ll \sum_{f \in H(p, \chi)} \frac{1}{(f, f)} h(V, t_f) \left| \sum_{l \leq L} x_l \lambda_f(l) \right|^2 \sum_{h \in H(q, \psi)} \frac{1}{(h, h)} h(W, t_h) |L(1/2, f \times h)|^2.$$

11. PUTTING IT ALL TOGETHER

Using the estimates from Sections 6, 7, 8, and 9, and letting $\theta = \theta_1 = \theta_2$ in (7.7) as well as (6.7), we get (putting explicitly in the orthonormal basis built into the trace formula)

$$(11.1) \quad \sum_{f \in H(p, \chi)} \frac{1}{(f, f)} h(V, t_f) \left| \sum_{l \leq L} x_l \lambda_f(l) \right|^2 \sum_{h \in H(q, \psi)} \frac{1}{(h, h)} h(W, t_h) |L(1/2, f \times h)|^2 \ll$$

$$\|x\|_2^2 \left((pq)^\epsilon L^\epsilon \{\text{delta-delta term}\} \right.$$

$$+ p^\epsilon q^{\epsilon-1} L^\epsilon \{\text{delta-Kloosterman term}\} + q^\epsilon p^{\epsilon-1} L^\epsilon \{\text{Kloosterman-delta term}\}$$

$$\left. + (pq)^{\theta-1+\epsilon} L^\epsilon + (pq)^{\theta-1/2+\epsilon} (1 + q^\delta) L^{1+\epsilon} \{\text{Kloosterman-Kloosterman term}\} \right).$$

Remark. (1) If we take $L = 1$, the trivial amplifier, we obtain the expected Lindelöf on average bound even with $\theta = 1/4$.

- (2) Remember the “abnormal” term $(1 + q^\delta)$ in the Kloosterman estimate comes from a contour shift to get the “main” term and “remainder” term in (6.25) and (6.26). The Kloosterman-Kloosterman bounds above in (11.1) come from these respective terms’ analysis in (6.62) and (7.6). We expect with more refined analysis and exploiting some kind of symmetry or functional equation, similar to what is done in [So], the “remainder” term is much smaller than the “main” term.
- (3) The term $(1 + q^\delta)$ being not symmetric in p and q is an artifact of that we are only amplifying the forms of level p in (11.1). If we put an amplifier on both forms $f \in H(p, \chi)$ and $h \in H(q, \psi)$ we would again expect a symmetry in the estimate.
- (4) Note in this paper, we do not apply any Ramanujan bounds towards the Fourier coefficients as all the analysis is done on the geometric side of the trace formula. The only crucial ingredient is the subconvexity bounds for GL_1 L-functions.

As mentioned, it is implied in the trace formulas used above that the f - and h -sums are orthonormal with respect to the Petersson inner product. This inner product also depends on the level p and q . Using that $(f, f) \ll p(\log p)^3$ and $(h, h) \ll q(\log q)^3$, and a choice of V, W as in section 10 to get the inequality (10.4) we get

$$\left| \sum_{l \leq L} x_l \lambda_{f^\circ}(l) \right|^2 |L(1/2, f^\circ \times h^\circ)|^2 \ll \|x\|_2^2 \left((pq)^{\theta+\epsilon} L^\epsilon + (pq)^{\theta+1/2+\epsilon} (1+q^\delta) L^{1+\epsilon} + (pq)^{1+\epsilon} L^\epsilon + p^{1+\epsilon} q^\epsilon L^\epsilon + q^{1+\epsilon} p^\epsilon L^\epsilon \right).$$

We now choose the coefficients as in [KMV],

$$(11.2) \quad x_l := \begin{cases} \lambda_{f^\circ}(l) & \text{if } l \text{ is prime } \leq L^{1/2} \\ -1 & \text{if } l \text{ is a square of a prime } \leq L^{1/2} \\ 0 & \text{otherwise.} \end{cases}$$

Now using the fact that

$$\left| \sum_{l \leq L} x_l \lambda_{f^\circ}(l) \right|^2 \gg (pq)^{-\epsilon} L$$

and

$$\|x\|_2^2 \ll L^{1/2+\epsilon},$$

we have

$$(11.3) \quad |L(f^\circ \times h^\circ, 1/2)| \ll (pq)^\epsilon \left((pq)^{\frac{\theta}{2}} L^{-1/4} + (1+q^\delta)^{1/2} (pq)^{\frac{\theta}{2}+\frac{1}{4}} L^{\frac{1}{4}} + \frac{(pq)^{1/2}}{L^{1/4}} + \frac{p^{1/2} + q^{1/2}}{L^{1/4}} \right).$$

We can simplify it to

$$(11.4) \quad |L(f^\circ \times h^\circ, 1/2)| \ll (pq)^\epsilon \left(\frac{(pq)^{\frac{\theta}{2}} + (pq)^{1/2} + p^{1/2} + q^{1/2}}{L^{1/4}} + (1+q^\delta)^{1/2} (pq)^{\frac{\theta}{2}+\frac{1}{4}} L^{\frac{1}{4}} \right).$$

If we choose $L = (pq)^{\frac{1}{2}-\theta}$ then (11.3) is bounded by

$$(pq)^\epsilon \left((pq)^{\frac{3\theta}{2}-1/8} + (pq)^{3/8+\frac{\theta}{4}} + p^{3/8+\frac{\theta}{4}} q^{\frac{\theta}{4}-1/8} + q^{3/8+\frac{\theta}{4}} p^{\frac{\theta}{4}-1/8} + (1+q^\delta) (pq)^{\frac{3}{8}+\frac{\theta}{4}} \right).$$

This completes Theorem 1.1.

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